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Modelling predictive validity problems: A partial identification approach

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Introduction

The anatomy of the predictive validity problem

Predictive validity is referred to the relations between test scores and any external variable to the test (American Educational Research Association et al., 2014). Statistical techniques that have been used for predictive validity include regression analysis and correlation coefficients between tests scores, X, and variables that are external to the test, Y (see, Pearson, 1903; Lawley, 1943; Guilliksen, 1950; Berry et al., 2013; Manzi et al., 2008; Technical Advisory Committee, 2010).

In the context of regression analysis, the conditional expectation of the external variable given the test scores, namely $\mathbb{E}(Y|X)$, is estimated. The predictive validity is assessed through the marginal effect, which quantifies the changes in this conditional expectation with respect to changes in the values of the test scores. This approach has been used in several predictive validity studies, for instance, GU et al. (2008) used the marginal effect to quantify the impact of scores in a patient satisfaction survey, X, on patients' medication adherence, Y. Moreover, a result in this study is that a positive marginal effect of patient satisfaction with pharmacist consultation service reflects a positive association between patient satisfaction and medication adherence. Thus, if changes in scores produces large (small) changes in the outcome of interest, then the effect of test scores will be high (low) on the outcome.

The predictive validity of a test can be evaluated in any field. For example, in Psychiatry, the Beck Depression Inventory (BDI, Beck et al., 1961) is used to evaluate depression symptoms. The score in the test gives information about depression level which is categorised into four levels of severity: Minimal, Mild, Moderate, and Severe. Green et al. (2015) studied the predictive validity of the BDI Suicide item through a Cox regression model where a conclusion of the study is that *the BDI suicide*

item significantly predicted both deaths by suicide and repeat suicide attempts. In educational measurement is common to evaluate the predictive validity of a selection university test (see for instance Meagher et al., 2006; Makransky et al., 2017; Kobrin et al., 2012). To assess the predictive validity of them is a statistical challenge because, although the test scores, X, are observed for all the applicants, the external variable to the test (the Graduate Point Average, GPA, for example), Y, is observed only in selected individuals. This problem is accordingly called *selection problem* and it arises when the random sampling process does not fully reveal the behaviour of the outcome on the support of the predictors (Manski, 1993).

From a statistical viewpoint the predictive validity of a selection test, under a regression approach, can be modelled as follows: let us define a binary random variable S such that $S = 1$ if the researcher observes the realisations of Y (e.g. $S = 1$ if researchers observe the GPA, at the first year in the University) and $S = 0$ if not. By using the Law of Total Probability (Kolmogorov, 1950), the conditional expectation is written as:

$$\mathbb{E}(Y|X) = \mathbb{E}(Y|X, S = 1)\mathbb{P}(S = 1|X) + \mathbb{E}(Y|X, S = 0)\mathbb{P}(S = 0|X)\,. \tag{1}$$

In equation (1), $\mathbb{E}(Y|X, S = 0)$ is impossible to be estimated because it is the conditional expectation of the non-observed outcomes. This fact implies a non-identification of $\mathbb{E}(Y|X)$. If $\mathbb{P}(S = 0|X) = 0$ (i.e. Y is fully observed), the conditional expectation is point identified; however, in the selection context this probability is not zero. Hence, the conditional expectation is not identified, and as a consequence the marginal effect is not identified either.

Researchers have devoted a great effort finding restrictions to identify $\mathbb{E}(Y|X)$. In what follows, the statistical strategies that have been used to tackle the identification problem are briefly explained to introduce the statistical approach in which the proposal is founded.

Tackling the problem

Weak ignorability assumption

Suppose the following identification restriction is imposed: $Y \perp S|X$[1] (see Imbens, 2000; Hirano and Imbens, 2004). To believe in this assumption is analogous to believing in that performance

[1] In words of Florens and Mouchart (1982), Y is conditionally orthogonal to S.

in the non-observed population is equal to the one in the observed population. Formally, this identification restriction asserts that

$$\mathbb{E}(Y|X, S = 0) = \mathbb{E}(Y|X, S = 1). \tag{2}$$

Equation (2) is a non-refutable assumption, which allow point-identifies $\mathbb{E}(Y|X)$ (Manski, 2007). In fact, it allows making inferences over the conditional expectation ignoring the non-observed values of the response variable, such that

$$\mathbb{E}(Y|X) = \mathbb{E}(Y|X, S = 1). \tag{3}$$

This approach is widely used in predictive validity studies. To give an example, in Geiser and Studley (2002) the predictive validity of the SAT scores, X, on the UC freshman grades, Y, was studied by using the full observed values of both X and Y only. However, because of information about the non-observed population is not taken into account, the consequence of using this identification restriction in the predictive validity of a selection test is that it could be underestimated (see Manzi et al., 2008).

The latent variable model

Suppose that we can identify $\mathbb{E}(Y|X)$ by assuming that

$$\begin{aligned} \mathbb{E}(Y|X) &= f_1(X) \\ \mathbb{E}(Y|X, S = 1) &= f_1(X) + f_2(X), \end{aligned}$$

for some functions f_1 and f_2. Suppose that the following specification is introduced: Y is observed if and only if $g(X) + U_2 > 0$, such that

$$\begin{aligned} Y &= f_1(X) + U_1; \quad \mathbb{E}(U_1|X) = 0 \\ S &= \mathbb{1}_{\{g(X)+U_2>0\}} \end{aligned}$$

where f_1 and g are real functions of X and (U_1, U_2) are non-observable random variables. Note that condition $\mathbb{E}(U_1|X) = 0$ allows to interpret $f_1(X)$ with respect to the sampling process as $\mathbb{E}(Y|X) = f_1(X)$, then:

$$\begin{aligned} \mathbb{E}(Y|X, S = 1) &= \mathbb{E}(f_1(X) + U_1|X, S = 1) \\ &= f_1(X) + \mathbb{E}(U_1|X, g(X) + U_2 > 0) \, . \end{aligned}$$

A particular case is Heckman's Selection Model (Heckman, 1976), where

$$\begin{aligned} \mathbb{E}(Y|X) &= X^\top \beta_1 \\ \mathbb{E}(Y|X, S=1) &= X^\top \beta_1 + \rho\sigma \frac{\phi(X^\top \alpha)}{\Phi(X^\top \alpha)} \,. \end{aligned}$$

Here, ϕ and Φ are the standard normal density and distribution functions, respectively, and $\rho\sigma = \mathbb{E}(U_1, U_2)$. For details of the estimators of both σ^2 and ρ_{XY} and its properties see Heckman (1979)[2]. In the selection tests context, an example is given in Kennet-Cohen et al. (1999) where the Heckman's approach was used for learning about the predictive validity of the Psychometric Entrance Test for Higher Education in Israel.

From a statistical perspective, imposing a restriction on f_1 implies a specification of the function that characterise the conditional expectation in the whole population, $\mathbb{E}(Y|X)$. Nevertheless, an assumption that restricts this conditional expectation can be incompatible with the reality (Manski, 2003).

It is well-known that statistical inference requires combine the data with assumptions about the population of interest. In fact, the logic of the inference is:

$$\text{Data} + \text{Assumptions} = \text{Conclusions}$$

(see Manski, 2013, p1). Thus, for a fixed data set: when assumptions change, conclusions can change dramatically. This fact motivate us to define assumptions about the non-observed values in order to draw more general conclusions for the population of interest. The main objective of this thesis is learning about both the conditional expectation and the marginal effect by making weaker assumptions than the standard ones, where it is assumed only one possible scenario: *the performance in the non-observed group is equal to the one in the observed group*. Our proposal is based on the partial identification approach, whose aim is to find a region to characterise the plausible solutions that are consistent with the belief the empirical researcher has about these parameters of interest in the non-observed population. In the partial identification literature, this region is defined by *identification bounds*, which are found bounding the non-observed parameter of interest.

[2] Marchenko and Genton (2012) proposed a similar approach on which a t distribution is used instead of a Normal distribution.

Partial identification approach

The partial identification approach was introduced by Manski (1989) in an influential paper entitled *The anatomy of the selection problem*. This dissertation is based on two main results of that paper. The first one is regarding the identification bounds for the conditional expectation, which are obtained as follows: Assuming that $Y \in [y_0, y_1]$ where $y_0 \leq y_1$, it follows that $y_0 \leq \mathbb{E}(Y|X, S = 0) \leq y_1$. By applying this inequality to equation (1) we obtain the identification region for $\mathbb{E}(Y|X = x)$, namely

$$\begin{aligned} \mathbb{E}(Y|X = x) \quad \in \quad & \Big[\mathbb{E}(Y|X = x, S = 1)\mathbb{P}(S = 1|X = x) + y_0\mathbb{P}(S = 0|X = x); \\ & \mathbb{E}(Y|X = x, S = 1)\mathbb{P}(S = 1|X) + y_1\mathbb{P}(S = 0|X = x)\Big] \end{aligned} \tag{4}$$

This region characterise all the plausible values for the conditional expectation in presence of non-observed outcomes by assuming that it is bounded by the range of Y. Note that the width of the bound is $(y_1 - y_0)\mathbb{P}(S = 0|X = x)$; thus the severity of the identification problem varies directly with the probability of non-observed outcomes (Manski, 2003). In the context of selection tests, it is expected that higher scores in the selection test produces high probabilities of observing the outcome. In contrast, lower scores produce low probabilities of observing the outcome. Thus, the severity of the identification problem is for lower scores in the selection test.

The second result that we use is regarding the identification bounds for marginal effects, which is accordingly computed by taking the derivative of $\mathbb{E}(Y|X)$ with respect to X. Note that by taking the derivative with respect to X in (1), we obtain that:

$$\begin{aligned} \frac{d\mathbb{E}(Y|X)}{dX} \quad = \quad & \frac{d\mathbb{E}(Y|X, S = 0)}{dX}\mathbb{P}(S = 0|X) + \mathbb{E}(Y|X, S = 0)\frac{d\mathbb{P}(S = 0|X)}{dX} \\ + \quad & \frac{d\mathbb{E}(Y|X, S = 1)}{dX}\mathbb{P}(S = 1|X) + \mathbb{E}(Y|X, S = 1)\frac{d\mathbb{P}(S = 1|X)}{dX}. \end{aligned} \tag{5}$$

In Equation (5), both $\mathbb{E}(Y|X, S = 0)$ and its derivative are not identified by the data generation process. The identification bounds for the marginal effect are obtained by combining the assumption of $y_0 \leq \mathbb{E}(Y|X, S = 0) \leq y_1$ with the assumption of that the non-observed derivative exists, such that

$$\left.\frac{d\mathbb{E}(Y|X, S = 0)}{dX}\right|_{X=x} \in [D_{0x}, D_{1x}].$$

Thus, by taking into account that in the selection context higher scores produce high probabilities to observe the outcome of interest (i.e. $\mathbb{P}(S=1|X)$ is an increasing function), it is obtained that:

$$\begin{aligned}\left.\frac{d\mathbb{E}(Y|X)}{dX}\right|_{X=x} \in \Bigg[& D_{0x}\mathbb{P}(S=0|X=x)+y_0\left.\frac{d\mathbb{P}(S=0|X)}{dX}\right|_{X=x} \\ & +\left.\frac{d\mathbb{E}(Y|X,S=1)}{dX}\right|_{X=x}\mathbb{P}(S=1|X)+\mathbb{E}(Y|X,S=1)\left.\frac{d\mathbb{P}(S=1|X)}{dX}\right|_{X=x}; \\ & D_{1x}\mathbb{P}(S=0|X=x)+y_1\left.\frac{d\mathbb{P}(S=0|X)}{dX}\right|_{X=x} \\ & +\left.\frac{d\mathbb{E}(Y|X,S=1)}{dX}\right|_{X=x}\mathbb{P}(S=1|X)+\mathbb{E}(Y|X,S=1)\left.\frac{d\mathbb{P}(S=1|X)}{dX}\right|_{X=x}\Bigg]. \quad (6)\end{aligned}$$

To illustrate the approach, and those currently used in the literature, let us consider a selection test with score scale from 150 to 850, and the GPA with scale from 1.0 to 7.0. In Figure (1) are shown the results for these three approaches. For the weak ignorability approach, $\mathbb{E}(Y|X,S=1)$ was estimated by using a Kernel regression as implemented in the `npreg` function from the `np` R-package (Hayfield and Racine, 2008). The Heckman's model was estimated with the `hekit` function from the `sampleSelection` R-package (Toomet and Henningsen, 2008). Regarding the identification bounds we used $y_0 = 1.0$ and $y_1 = 7.0$ (the minimum and maximum possible GPA in this example). The estimation of $\mathbb{E}(Y|X,S=1)$ under the weak ignorability approach was used. The probability to observe Y, $\mathbb{P}(S=1|X)$, was estimated by using the Probit model (Bliss, 1934).

From Figure (1), it can be seen that different conclusions can be drawn although the data have not changed. Note that under the weak ignorability assumption there is a positive relationship between test scores and the GPA; hence the marginal effect will be a constant function of the scores. From the Heckman's viewpoint, this relationship is not linear. Moreover, there is a quadratic relationship between test scores and the GPA, and therefore the marginal effect will be a non-constant function of the test scores. When the identification bounds are analysed, assuming that the GPA is bounded (i.e. $1.0 \leq \text{GPA} \leq 7.0$) we can observe that the severity of the identification problem is for lower scores. In fact, for selection tests it is expected that for lower scores $\mathbb{P}(S=0|X) \to 1$. In contrast, for higher scores it is expected that $\mathbb{P}(S=0|X) \to 0$. Hence, for high scores, the width of the bounds tend to be the regression line under the ignorability assumption, as well as it is reflected in Figure (1). Only assuming that $E(Y|X,S=0)$ is bounded by the range of Y, identification

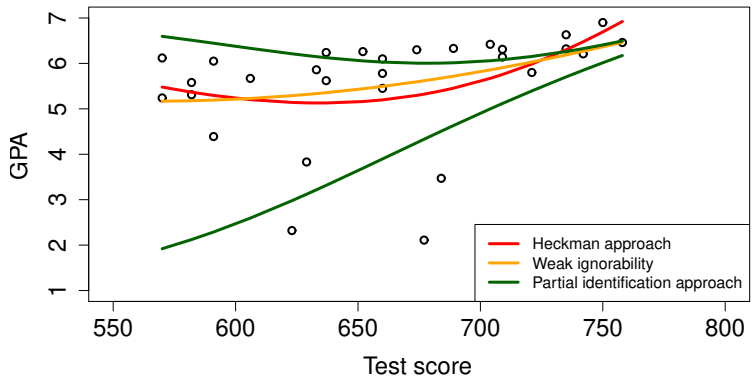


Figure (1) Regression under Weak ignorability, Regression under Heckman approach, and Identification bounds for the regression.

bounds give us information about all the plausible solutions for the regression of the GPA on the test scores.

Note that the regression under weak ignorability, for all test scores, is a plausible solution for learning about the conditional expectation (it is in between of the bounds). However, the Heckman's model is a non-plausile solution for learning about the conditional expectation when higher scores are considered. Throughout the document we will give an interpretation of the bounds in the predictive validity of selection tests context. Despite of we can compute identification bounds for the conditional expectation by using the range of Y only, the non-observed derivative is not necessarily bounded. Thus, we need to impose identification restrictions by using any criteria which will allow establish expressions for both D_{0x} and D_{1x} in the identification region given in Equation (6).

In the partial identification literature there are not results related to establish identification restrictions bounding the derivative of the conditional expectation in the predictive validity context. Thus, this dissertation intends to add to the literature on partial identification by discussing how to establish identification restrictions by using few general ideas about the problem that a researcher wants to model.

Outline of the dissertation

The organisation of the dissertation is as follows: in Chapter 1 we intend to set up identification restrictions for the non-observed marginal effect which are based on a desired property of the selection tests: *higher scores on the test would translate in better performance at higher education*. Nevertheless, the tackled problem in Chapter 1 is based on that the available information comes from a population that is partitioned into two subpopulations: *The outcome is observed*, and *the outcome is not observed*. Then, a natural extension is when the available information comes from a population that is partitioned into G groups, where each of them has partial observability of the outcome. In this context, in Chapter 2 is defined a new way to analyse and interpret the marginal effect in a partitioned population. The interpretation considers not only the marginal effect in each group but also the differences in predicted outcomes and the size of the groups. In Chapter 3 we incorporate the partial observability of the outcome in each group. At this point, we extend the identification regions given in (4) and (6) to the case of multiple groups, where each of them have different patterns of missing outcomes. Identification restrictions are based on the beliefs about the considered selection system. The dissertation ends in Chapter 4 where general conclusions about the methodology are drawn. Additionally, a generalisation of the proposal is described as a future work.

Final considerations

The dissertation is a collection of manuscripts that are either published/in press or there are ready for submission. Hence, there might be some overlapping among the Chapters.

The chapters correspond to the following original manuscripts:

Chapter 1: It is partially based on: Alarcón-Bustamante, E., San Martín, E. & González, J. (2020). *Predictive validity under partial observability*. In Wiberg, M., Molenaar, M., González, J., Böckenholt, U., and Kim, JS. (Eds.), Quantitative Psychology. IMPS2019. Springer Proceedings in Mathematics & Statistics, vol 322. Springer, Cham. DOI 10.1007/978-3-030-43469-4_11

Chapter 2: It is fully based on: Alarcón-Bustamante, E., San Martín, E. & González, J. (in

press). *On the marginal effect under partitioned populations: Definition and Interpretation*. In Wiberg, M., Molenaar, M., González, J., Böckenholt, U., and Kim, JS. (Eds.), Quantitative Psychology. IMPS2020. Springer Proceedings in Mathematics & Statistics.

Chapter 3: Alarcón-Bustamante, E., San Martín, E. & González, J. *On the marginal effect under partially observed partitioned populations: a functional predictive validity coefficient* (To be submitted).

Chapter 1

Learning about the predictive validity under partial observability

1.1 Introduction

A test is used to learn about a behaviour of interest. The relationship between test scores and any variable external to the test may be used to predict some (future) behaviour of the individuals tested (Lord, 1980) in the sense that we are interested in the conditional distribution of those external variables given test scores. We focus our attention on tests that are used in a selection process, specifically on admission to the higher education. The purpose of the test is to select the "best applicants" in some specific sense which is typically operationalized through a cut-off score. It is supposed that the cut-off is defined in such a way that higher scores on the test would translate in better performance at higher education.

In this context, it is necessary to assess and measure the quality of the selection, which leads to analyse the *validity* and *reliability* of the admission test. Regarding validity, the American Educational Research Association et al. (2014) define it as *the degree to which evidence and theory support the interpretations of test scores for proposed uses of tests*. In particular, the predictive validity of a test is defined as the evidence based on relations to other variables: in an admission test, these variables are supposed to be chosen according to the selection purposes of a higher educational system. Following this definition, the analysis of the relationship between test scores and

any external variable to the test provide an important source of predictive validity evidence.

To assess the predictive validity of a selection test is a challenge because the outcome measured at higher education is observed only in the selected group, whereas the scores of the selection test are observed for the whole population of applicants. This problem is accordingly called *selection problem* and arises when the sampling process does not fully reveal the behavior of the outcome on the support of the predictors (Manski, 1993).

Statistical procedures used for the evaluation of the predictive validity include regression models with truncated distributions (Nawata, 1994; Heckman, 1976, 1979; Marchenko and Genton, 2012) and corrected Pearson correlation coefficient (Thorndike, 1949; Pearson, 1903; Mendoza and Mumford, 1987; Lawley, 1943; Guilliksen, 1950). In the context of admission university selection tests, a common practice to evaluate the predictive validity of the selection tests is to measure the correlation between the obtained scores and the cumulative grade point average (GPA) at the first year of the students in the university.

Although those procedures constitute solutions to the problem of learning about the predictive validity, it is explicitly assumed a prior knowledge for the performance of the *whole population*[1], that is, it is assumed that the conditional distribution of the outcome given the scores is known up to some parameters. However, we argue that this assumption is not pertinent because the consequence of the partial observability is that the conditional distribution of the outcome given the scores is not identified and therefore assuming any structure for the non-observed group could not be assessed empirically (Manski, 1993). For this reason, this approach does not solve satisfactory the problem of predictive validity. In the educational measurement literature, the predictive validity is typically analysed through the *marginal effect* (for instance see Leong, 2007; Goldhaber et al., 2017; Geiser and Studley, 2002), that is, the derivative of the conditional expectation of the outcome given scores, with respect to the scores. However, as the conditional expectation is not identified, the marginal effect is not identified either.

Thus, we propose a methodological approach that allows to learn about the predictive validity of selection tests through the marginal effects, under partial observability of the outcome. We use a partial identification approach in order to define a region that characterises the set of all admissible

[1]The term whole population refers to the population that is integrated by two subpopulations: the population where the outcome is observed and the one where the outcome is not observed (from here on the observed group and the non-observed group, respectively).

values for the marginal effects. This region is delimited by identification bounds. This approach works if explicit assumptions on the unobservable distributions are made, the idea being that such assumptions be weaker than the standard ones above-mentioned (Manski, 2013). We propose to find identification bounds by assuming that the selection test is such that higher scores would translate to higher values of the outcome, i.e., it is considered that there is a positive relationship between test scores and the outcome. This assumption reflects an optimistic viewpoint on the selection test and the idea is to get conclusions to be compared with other perspectives. Identification bounds are rigorously operationalized through the monotonicity of the conditional expectation of the outcome given the test score.

The general framework of the partial identification approach is introduced in Section 1.2. In section 1.2.1 the partial identification framework of the conditional expectation is formalised. Identification bounds for marginal effects are formally described in Section 3.3. In Section 1.2.3 the identification bounds for the marginal effects in the selection problem context are formally characterised. In Section 1.3, the performance of the proposed methodology is illustrated on a real data set from the selection test used in the university admission Chilean system. Conclusions and further work are discussed in Section 1.4.

1.2 Partial identification framework

1.2.1 Partial identification of the conditional expectation

Let Y denote the outcome variable, X a test score, and S a binary random variable with $S = 1$ if the outcome is observed and $S = 0$ otherwise. Consequently, each member of the population is characterized by a triple (Y, S, X). We focused our attention on the conditional expectation of the outcome Y given a test score X. By the Law of Total Probability (Kolmogorov, 1950), it follows that

$$\mathbb{E}(Y|X) = \mathbb{E}(Y|X, S = 1)\mathbb{P}(S = 1|X) + \mathbb{E}(Y|X, S = 0)\mathbb{P}(S = 0|X) \, . \tag{1.1}$$

In (1.1), $\mathbb{E}(Y|X, S = 1)$, $\mathbb{P}(S = 1|X)$ and $\mathbb{P}(S = 0|X)$ are identified by the data generating process. However, $\mathbb{E}(Y|X, S = 0)$ is not identified. Consequently $\mathbb{E}(Y|X)$ is not identified either.

One solution for this problem is to assume *weak ignorability*, namely $Y \perp S|X$[2], which implies

[2]In words, $Y \perp S|X$ indicates that Y is conditionally orthogonal to S (see Florens and Mouchart, 1982)

that $\mathbb{E}(Y|X) = \mathbb{E}(Y|X, S = 1)$. The assumption of weak ignorability allows making inferences on $\mathbb{E}(Y|X)$ *ignoring* the non-observed values of Y, which can lead to underestimation of the predictive capacity of the selection test (Manzi et al., 2008).

Assuming that $Y \in [y_0, y_1]$ where y_0 and y_1 are the minimum and the maximum possible GPA, respectively, it follows that $y_0 \leq \mathbb{E}(Y|X, S = 0) \leq y_1$. By applying this inequality to equation (1.1) we have

$$\begin{aligned}\mathbb{E}(Y|X, S = 1)\mathbb{P}(S = 1|X) + y_0\mathbb{P}(S = 0|X) &\leq \mathbb{E}(Y|X) \\ &\leq \mathbb{E}(Y|X, S = 1)\mathbb{P}(S = 1|X) + y_1\mathbb{P}(S = 0|X).\end{aligned}$$

The lower bound of the conditional expectation is interpreted as the value $\mathbb{E}(Y|X)$ takes if, in the non-observed group, Y is always equal to y_0 (i.e., if all students obtained the worst GPA). Regarding the upper bound, it is interpreted as the value $\mathbb{E}(Y|X)$ takes if, in the non-observed group, Y is always equal to y_1 (i.e., if all students obtained the best GPA) (Manski, 1989).

1.2.2 Partial identification of the marginal effects

As it is well known, the marginal effect is defined as

$$ME^X = \frac{d\mathbb{E}(Y|X)}{dX}\,.$$

Figure (1.1) shows how variations in X reflect in variations on $\mathbb{E}(Y|X)$. These variations are quantified by the change of $\mathbb{E}(Y|X)$ with respect to changes in X, defined by ME^X. The blue dash lines represent the marginal effect evaluated at the point $X = x$. From equation (1.1) it follows that,

$$\begin{aligned} ME^X &= \left(\mathbb{P}(S = 0|X)M.E^X_{S=0} + \mathbb{P}(S = 1|X)M.E^X_{S=1}\right) \\ &+ \left([\mathbb{E}(Y|X, S = 1) - \mathbb{E}(Y|X, S = 0)]\,\frac{d\mathbb{P}(S = 1|X)}{dX}\right)\,, \end{aligned} \tag{1.2}$$

where $ME^X_{S=s}$ is the marginal effect of X over the group $S = s$, with $z \in \{0, 1\}$.

As it was mentioned in Section 2.1, the conditional expectation is not identified in the context of the selection problem. Consequently, the marginal effect will not be identified either. In fact, in equation (1.2) both $\mathbb{E}(Y|X, S = 0)$ and $ME^X_{S=0}$ are not identified by the sampling process.

In order to find the identification bounds for the marginal effects, suppose that $D_{0x} \leq ME^{X=x}_{S=0} \leq D_{1x}$, where $M.E^{X=x}_{S=s}$ is the marginal effect of X over the group $S = s$ evaluated at $X = x$. Thus,

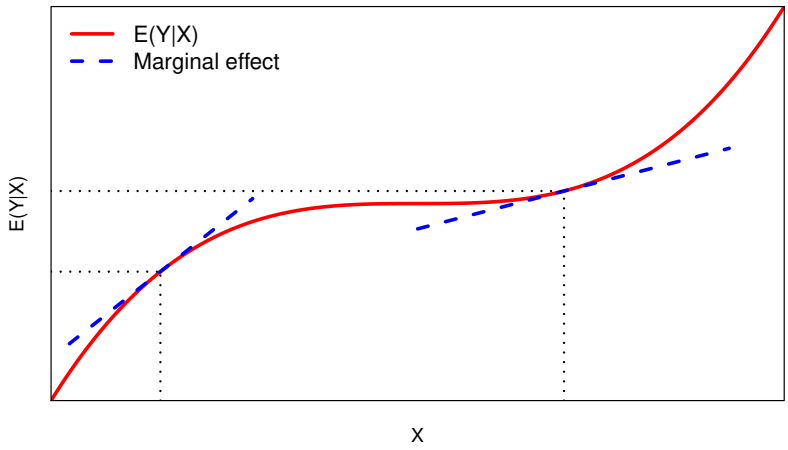


Figure (1.1) The regression function and the marginal effect

we assume that the marginal effect for the non selected population exist, which means that if this population had been selected, the score of the selection test would have predicted the outcome with an associated marginal effect. This marginal effect is subject to uncertainty, as reflected by the bounds, which depends on specific values of X.

Considering the above assumption and that $y_0 \leq \mathbb{E}(Y|X, S = 0) \leq y_1$, ME^X evaluated at $X = x$, denoted by $ME^{X=x}$, is bounded as follows

$$D_{0x}\mathbb{P}(S = 0|X = x)+\mathbb{P}(S = 1|X = x)ME_{S=1}^{X=x}+$$
$$[\mathbb{E}(Y|X = x, S = 1) - y_0] \left.\frac{d\mathbb{P}(S = 1|X)}{dX}\right|_{X=x} \leq ME^{X=x} \leq$$
$$D_{1x}\mathbb{P}(S = 0|X = x)+\mathbb{P}(S = 1|X = x)ME_{S=1}^{X=x}+$$
$$[\mathbb{E}(Y|X = x, S = 1) - y_1] \left.\frac{d\mathbb{P}(S = 1|X)}{dX}\right|_{X=x}$$

Note that an assumption on $\mathbb{E}(Y|X, S = 0)$ does not by itself restrict the marginal effect, but an assumption on both $\mathbb{E}(Y|X, S = 0)$ and the marginal effect for the non-observed group, does (Manski, 1989).

1.2.3 Identification bounds for marginal effects

According to Manski (2003, 2007, 2005), researchers sometimes take credible information about properties of the outcome. For example, there might be reasons to believe that the outcome increase/decrease monotonically when the predictor increase/decrease. Thus, in a University Admission System it can be assumed that, from an optimistic view point, a higher score in the selection test implies a higher GPA at the first year of the university. What can be concluded for the marginal effect under this optimistic assumption? The partial identification analysis aims to answer this question.

Selection tests are used to select the best applicants such that higher scores, X, would imply higher values of the outcome, Y. This fact allows to think that the conditional expectation of Y given X is a non decreasing function of X and, consequently, the marginal effect in the non-observed group should be be greater or equal than zero, i.e., $ME_{S=0}^{X=x} \geq 0$. Under this assumption, it is clear that $D_{0x} = 0$, and therefore identification bounds for the marginal effect are given by:

$$\mathbb{P}(S=1|X=x)ME_{S=1}^{X=x} + [\mathbb{E}(Y|X=x,S=1) - y_0] \left.\frac{d\mathbb{P}(S=1|X)}{dX}\right|_{X=x} \leq ME^{X=x} \leq$$
$$D_{1x}\mathbb{P}(S=0|X=x) + \mathbb{P}(S=1|X=x)ME_{S=1}^{X=x} + [\mathbb{E}(Y|X=x,S=1) - y_1] \left.\frac{d\mathbb{P}(S=1|X)}{dX}\right|_{X=x}$$

For D_{1x}, suppose that the predictability of X over Y in the non-observed group can not be higher than the maximum observed marginal effect. This assumption is realistic because one objective of selection tests is to choose those applicants that would obtain a better outcome than those who were not selected. In terms of and identification restriction for the non-observed marginal effects, this assumption translates to $D_{1x} = \max_{x \in X} \left\{ M.E_{S=1}^{X=x} \right\}$.

1.3 Illustration

The evolution of the university admission system in Chile includes the *baccalaureate* test, administrated during 1931 and 1966; and the *Prueba de Aptitud Académica* (PAA, for their initials in Spanish), administered during the period 1967 - 2002. These tests were criticised, among other reasons, because of their low predictive capacity (Grassau, 1956; DEMRE, 2016; Donoso, 1998).

Since 2004 the selection process is partially based[3] on scores from the *Prueba de Selección Universitaria* (PSU, for their initials in Spanish). The PSU is elaborated based on the secondary school curriculum and includes two mandatory tests (Mathematics and Language and communication), and two elective tests (Sciences and History, Geography and Social Sciences). According to Donoso (1998), one of the reasons for the evolution of the Chilean university admission system is the necessity to increase predictive capacity of the selection tests.

To illustrate the partial identification approach proposed, we analyse the predictive validity of the mandatory PSU tests over the GPA of students in the first year in a Chilean university. It is important to highlight that the analysis is based on the top one university[4], so that the assumption that the performance of non-enrolled students will be at most equal to that of enrolled students, is tenable.

1.3.1 Estimation of the identification bounds

Let Y denote the GPA and X the score in the selection factor of interest. The conditional expectation $\mathbb{E}(Y|X, S = 1)$ was estimated by an adaptive local linear regression model using a symmetric Kernel as implemented in the `loess.as` function from the `fANCOVA` R-package (Wang, 2010). The probability of being observed was modeled assuming that $\mathbb{P}(S = 1|X) = \Phi(\alpha X)$, where $\Phi(\nu)$ is the standard normal cumulative probability distribution evaluated at ν. We considered the standardised values for both X and Y. This means that, by taking into account that in Chile a score of 1 is the minimum GPA that could be obtained, and a score of 7 the maximum one, we evaluated our method using $y_0 = \frac{1-\overline{GPA}}{sd(GPA)}$ and $y_1 = \frac{7-\overline{GPA}}{sd(GPA)}$, where $\overline{GPA}$ is the mean of the observed GPA (5.042) and $sd(GPA)$ is its standard deviation (0.568).

1.3.2 Results

Figures 1.2a, 1.2b and show the identification bounds of the marginal effect for both the Mathematics test and the Language and Communication test. Our method is compared with the marginal effect of the multiple linear regression model, the traditional procedure that have been used in Chile

[3] Additional to these tests there are other selection factors that are considered in the selection process, namely Ranking and High school grade point average (NEM)

[4] According to the Quacquarelli Symonds University Rankings 2019.

in order to evaluate the predictive capacity of the selection factors. For this model, the marginal effect of X is given by its regression coefficient.

For the Mathematics test, it can be seen that the marginal effects are not necessarily a constant function of the test score. In this case, a higher score produces a higher marginal effect. In other words, a good performance in the Mathematics test stands for a better performance in the GPA. Note that these type of conclusions can not be obtained using the traditional procedure. Regarding the marginal effects for the Language and Communication's test, contrary to what was seen for the Mathematics test, the identification bounds are nearly a constant function of the test score. This means that a good performance in the Language and Communication's test does not stand for a better performance in the GPA at the first year.

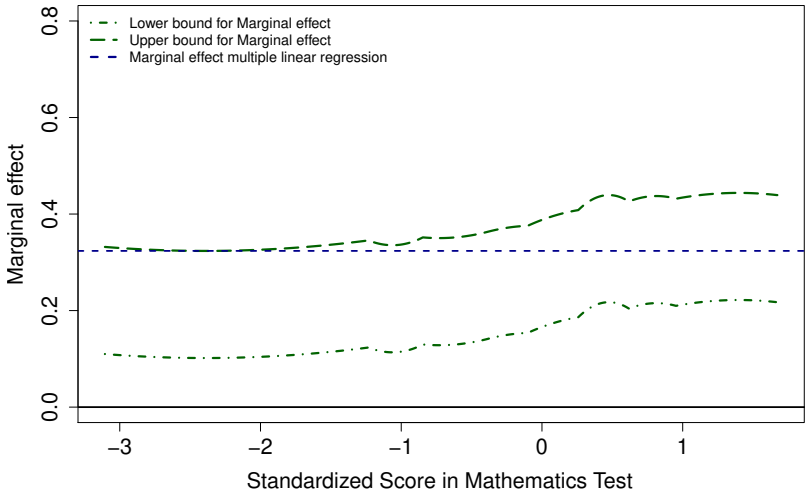
If a policymaker is ready to believe in these hypothesis, it can be concluded that the marginal effect computed under ignorability is coherent with them, as well as the case in the Mathematics test (see Figure 1.2a). In contrast, the multiple linear regression model does not allow to conclude the marginal effect of the Language and Communication test over the GPA because its marginal effect is out of the bounds (see Figure 1.2b). Hence, it is a not a plausible solution under the identification restrictions. Moreover, considering that the marginal effect in this test is lower than the lowest bound, we can conclude that the current methodological strategy used in Chile is a non-pertinent solution in the Language and Communication test case

As it was mentioned before, the identification bounds were computed under an optimistic scenario of the selection process. If we assume this scenario for the evaluated undergraduate program, and a linear regression model is used to extract conclusions about the marginal effect of these selection tests, it is better to use the Mathematics test instead the Language one.

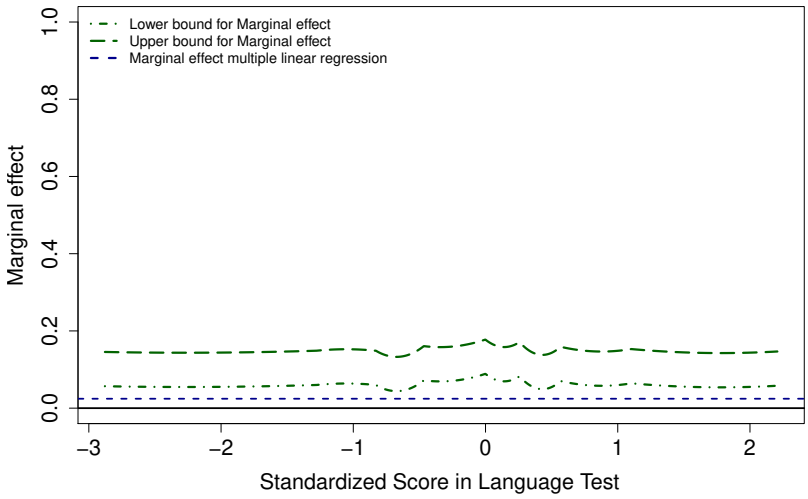
1.4 Conclusions and Discussion

We have presented a method that allows to learn about the predictive validity of selection tests through the marginal effect under partial observability.

Our partial identification-type solution characterises the set of all admissible values of the marginal effect. i.e., if the proposed model for the evaluation of the predictive capacity captures the information "the performance of the non-observed group is at most equal to the performance of the observed group", then the marginal effect of X must lie between the identification bounds.



(a) Identification bounds for marginal effect in Mathematics test.



(b) Identification bounds for marginal effect in Language and Communication test.

Figure (1.2) Identification bounds for the marginal effect of both Mathematics and Language Test

Although other approaches have been proposed to tackle the selection problem by assuming that the regression line does not change between the observed group and the non-observed group, our proposal has the advantage of not assuming any parametric structure for the non-observed group, as we only use properties of the selection tests. More specifically, we used monotonicity assumptions in order to find the set of all the possible values of the marginal effect by considering that the selection process is correct. However, this scenario make sense only when information about only one group with partial observability of the outcome is available. Extending the approach for the scenario where information about more than one groups is available is a topic that is analysed in the next Chapters.

Chapter 2

On the marginal effect under partitioned populations: Definition and Interpretation

2.1 Introduction

In social sciences and other fields, the impact that an exogenous explanatory random variable, X, (e.g., the score of a selection test) has on the outcome random variable, Y, (e.g., the cumulative grade point average, GPA) is usually measured through the marginal effect (see Geiser and Studley, 2002; DEMRE, 2016; Manzi et al., 2008; Grassau, 1956). The marginal effect quantifies the changes in the conditional expectation with respect to changes in the values of X: if changes in X produces large (small) changes in Y, then the effect of X will be high (low) on Y.

In predictive validity studies involving university selection tests, one of the main goals is to characterise the marginal effect taking into account that the population of interest is partitioned in groups or clusters (universities, countries, sex, among others). In this context, the conditional expectation of the GPA is conditioned not only on the test score, but also on a random variable, Z, characterising the groups. The most common approach is to use a multiple linear regression model with interaction terms between the test score and the group variable Z. By taking the difference between the marginal effect of a group of interest and one of reference, researchers compare the

impact of the test scores on the GPA between groups. As a matter of fact, the effect that X has on the group z with respect to the reference, z', corresponds to the "interaction effect", which is quantified by the corresponding interaction regression coefficient (see Cameron and Trivedi, 2005; Cornelissen, 2005; Powers and Xie, 1999; Norton et al., 2004; Ai and Norton, 2003; Long and Mustillo, 2018).

Formally, researchers are learning about the marginal effect of X on Y in a partitioned population by using the regression $\mathbb{E}(Y|X,Z)$, typically a linear one with some interaction terms. Thus, the analysis is separately made for each group while the interest is to report and draw conclusions based on a global analysis. As an example, in Miller and Frech (2000) a regression analysis is used to determine the effect of each explanatory variables on life expectancy measures and infant mortality for 21 OECD countries. Among the explanatory variables, the authors consider pharmaceutical consumption indexes, per capita income and other lifestyle factors such as tobacco use, alcohol consumption and richness of diet. Their study focuses on both a global analysis, reporting the effects of the explanatory variables on the outcomes, and an analysis by group, reporting the marginal effect of some explanatory variables for four countries (France, Italy, US and Ireland). In this chapter, we will show that this type of analysis needs to be carefully improved, the motivation being that a trend can appear when different groups are analysed separately, and possibly disappear when they are combined. This phenomenon is related to the Simpson's Paradox (see Simpson, 1951; Blyth, 1972).

As a matter of fact, we combine the groups through the Law of Total Probability, which lead to define $\mathbb{E}(Y|X)$ as a mixture of the corresponding conditional expectations for each group with mixing weights depending on each group. Thus, we compute the marginal effect of X on Y by using $\mathbb{E}(Y|X)$, instead of $\mathbb{E}(Y|X,Z)$. We define this marginal effect as the *Global Marginal Effect*, which is interpreted as the total marginal effect for partitioned populations. Although from this result it might be intuitive that the global marginal effect is obtained as a convex combination of the marginal effects for each group, we show that an additional term that depends on the predictive outcomes Y's by X's is also included in the definition.

The rest of the chapter is organised as follows. In Section 2.2 the concept of *global marginal effect* is formally defined and its properties are discussed. A detailed analysis of the function that characterise the global marginal effect is also presented in this section. An illustration showing the use of the global marginal effect in a real data set is presented in Section 2.3. The chapter ends in

Section 3.5 drawing conclusions and with a discussion.

2.2 Global Marginal Effect

2.2.1 Definition of the global marginal effect

Let us consider a population that is partitioned in groups or clusters and for which score data (X, Y) are observed. Let Z be a categorical random variable such that $Z = z$ if the statistical unit belongs to the group z for $z \in \{0, \ldots, G\}$. Thus, each member of the population is characterised by a triple (Y, X, Z). By applying the Law of Total Probability, the conditional expectation, $\mathbb{E}(Y|X)$, for the full population is obtained as

$$\mathbb{E}(Y|X) = \sum_z \mathbb{E}(Y|X, Z = z)\mathbb{P}(Z = z|X). \tag{2.1}$$

Equation (2.1) provides a global and unique conditional expectation function for the population, which contains the information of all the groups. In particular, this function could be characterised as a global model composed of different regression models (one for each group). The component models are those that relate the scores variables for each group, and a model for the categorical variable Z. The *Global Marginal Effect* is accordingly obtained by taking the derivative with respect to X in Equation (2.1), namely

$$\frac{d\mathbb{E}(Y|X)}{dX} = \sum_z \frac{d\mathbb{E}(Y|X, Z = z)}{dX}\mathbb{P}(Z = z|X) + \sum_z \mathbb{E}(Y|X, Z = z)\frac{d\mathbb{P}(Z = z|X)}{dX}. \tag{2.2}$$

From (2.2), it can be seen that the global marginal effect is not only the weighted average of the marginal effects in $Z = z$, but it also depends on the marginal effects observed through the categorical regression, $\mathbb{P}(Z = z|X)$. In particular, it follows that if $Z \perp\!\!\!\perp X$, equation (2.2) reduces to

$$\frac{d\mathbb{E}(Y|X)}{dX} = \sum_z \left[\frac{d\mathbb{E}(Y|X, Z = z)}{dX}\right] \mathbb{P}(Z = z).$$

Thus, the marginal effect in partitioned populations is a weighted average of the marginal effects in $Z = z$, if belonging to the group Z does not depend on X.

For the case when $Z \not\perp\!\!\!\perp X$, and taking into account that $\sum_z P(Z = z \mid X) = 1$, the global marginal effect in Equation (2.2) can be rewritten as follows:

$$\begin{aligned}\frac{d\mathbb{E}(Y|X)}{dX} &= \sum_z \frac{d\mathbb{E}(Y|X, Z = z)}{dX}\mathbb{P}(Z = z|X) + \\ &+ \sum_{z \neq z'} [\mathbb{E}(Y|X, Z = z) - \mathbb{E}(Y|X, Z = z')]\frac{d\mathbb{P}(Z = z|X)}{dX}; \end{aligned} \tag{2.3}$$

here z' is the label of a reference group. It can be verified that the global marginal effect is invariant under the chosen reference group; for a proof, see Appendix A.

Equation (2.3) corresponds to the sum of two functions, namely

$$a(X) = \sum_z \frac{d\mathbb{E}(Y|X, Z = z)}{dX}\mathbb{P}(Z = z|X), \text{ and}$$

$$b(X) = \sum_{z \neq z'} [\mathbb{E}(Y|X, Z = z) - \mathbb{E}(Y|X, Z = z')]\frac{d\mathbb{P}(Z = z|X)}{dX},$$

where $a(X)$ is a convex combination of the marginal effects in each group with weights being a function of X. Then, $a(X)$ will vary according to the variations of the weights as a function of X. The term $b(X)$ is the sum of the differences of the predicted Y, multiplied by the marginal effect of X on Z.

In the next section, both functions $a(X)$ and $b(X)$ are analysed when the population is assumed to be partitioned in three groups. A linear and a multinomial logistic regression model are considered for $\mathbb{E}(Y|X, Z = z)$ and $\mathbb{P}(Z = z|X)$, respectively.

2.2.2 Interpretation of the global marginal effect

Let us consider three groups, (i.e., $z \in \{0, 1, 2\}$). By using the invariant property of the global marginal effect, without loss of generality we take $z' = 0$ as the reference group, then

$$\begin{aligned}\frac{d\mathbb{E}(Y|X)}{dX} &= \sum_{z=0}^{2} \frac{d\mathbb{E}(Y|X, Z = z)}{dX}p_z(X) + \\ &+ \sum_{z=1}^{2} [\mathbb{E}(Y|X, Z = z) - \mathbb{E}(Y|X, Z = 0)]\frac{dp_z(X)}{dX},\end{aligned}$$

where $p_z(X) = \mathbb{P}(Z = z|X)$. If a linear function for $\mathbb{E}(Y|X, Z = z)$ is considered (i.e., $\mathbb{E}(Y|X, Z = z) = \delta_z + \gamma_z X$), the marginal effect is a constant function of X, namely γ_z. On the other hand, if a multinomial logistic function $F(u_z) = \exp\{u_z\}/(1 + \sum_{j=1}^{2} \exp\{u_j\})$, $u_z = \alpha_z + \beta_z X$ and $z \in \{1, 2\}$, is used for the prediction function, $p_z(X)$, the marginal effect of X on Z is given by

$$\frac{dp_z(X)}{dX} = \begin{cases} p_z(X)\left(\beta_z - \sum_{j=1}^{2} \beta_j p_j(X)\right) & \text{if } z \in \{1, 2\} \\ -\sum_{z=1}^{2} p_z(X)\left(\beta_z - \sum_{j=1}^{2} \beta_j p_j(X)\right) & \text{if } z = 0 \end{cases}$$

(See Wooldridge, 2010; Greene, 2003). This marginal effect inform us about the change in predicted probabilities due to the changes in X (Wulff, 2014).

Analysis of $a(X)$

Note that:

$$a(X) = \sum_{z=0}^{2} \gamma_z p_z(X),$$

which corresponds to a mixture of γ_z's with mixing weights defined by $p_0(X), p_1(X)$, and $p_2(X)$.

For ease of exposition, let us consider the following particular case as an example

$$\mathbb{E}(Y|X, Z = 1) \geq \mathbb{E}(Y|X, Z = 0); \quad \text{and} \quad \mathbb{E}(Y|X, Z = 2) \geq \mathbb{E}(Y|X, Z = 0),$$

and $\gamma_0 > \gamma_2 > \gamma_1$. The group 0 has the lowest predicted Y for all the values of X, but its marginal effect, γ_0, is higher than both γ_1 and γ_2. Hence, its "importance" in $a(X)$ will depends on how $p_0(X)$ varies with respect to X. This case is graphically illustrated in Figure 2.1. From the right-side panel in Figure 2.1, it can be seen that $p_0(X)$ is a decreasing function of X (i.e., for higher values of X, a lower probability of belonging to the group 0 is found), then for lower values of X, $a(X)$ is influenced by $\gamma_0 p_0(X)$. In this sense, $a(X)$ can be interpreted as a trade-off among the marginal effects of the groups: as a function of X, it depends not only on the highest value γ_z for a specific group z, but also on the size of such a group.

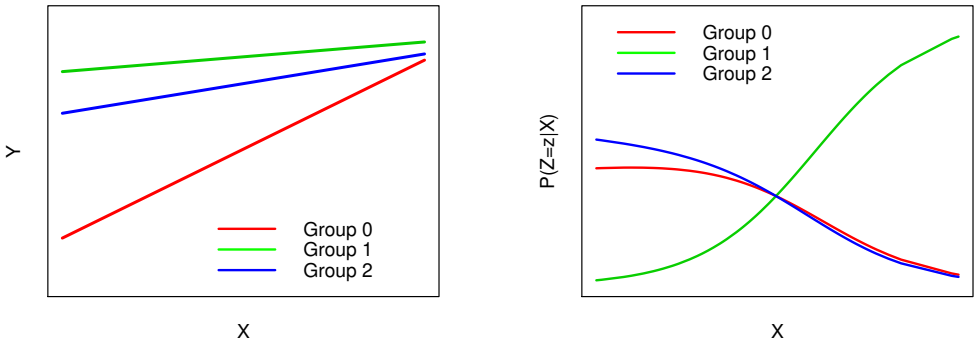


Figure (2.1) Example situation. The left-side panel shows $\mathbb{E}(Y|X, Z = z) = \delta_z + \gamma_z X$. The right-side panel shows $p_z(X) = F(u_z)$ for $z \in \{1, 2\}$, and $p_0(X) = 1 - \sum_{z=1}^{2} p_z(X)$.

Analysis of $b(X)$

In our example,

$$b(X) = \sum_{z=1}^{2} [(\delta_z - \delta_0) + (\gamma_z - \gamma_0)X] \left(p_z(X)[\beta_z(1 - p_z(X)) - \beta_j p_j(X)]\right),$$

with $j \neq z$. Let us analyse the first component of $b(X)$, namely

$$b_1(X) = [(\delta_1 - \delta_0) + (\gamma_1 - \gamma_0)X] \left(p_1(X)[\beta_1(1 - p_1(X)) - \beta_2 p_2(X)]\right).$$

When

$$\mathbb{E}(Y|X, Z = 1) \geq \mathbb{E}(Y|X, Z = 0),$$

$b_1(X)$ will increase (or decrease) according to

$$p_1(X)[\beta_1(1 - p_1(X)) - \beta_2 p_2(X)],$$

which can be written as

$$p_1(X)[\beta_1 p_0(X) - (\beta_2 - \beta_1)p_2(X)]. \tag{2.4}$$

Note that Equation (2.4) depends not only on the probability of belonging to the group 1, but also on the probability of belonging to the group 0 and 2. In this context, if x_1^* is the inflection point of

$p_1(X)$, which is a monotonic increasing function of X, then for all $x > x_1^*$

$$\mathbb{P}(Z = 1|X = x) > \mathbb{P}(Z \neq 1|X = x).$$

Thus, for all $x > x_1^*$, $b_1(X)$ is influenced by $\mathbb{P}(Z = 1|X = x)$. In contrast, for all $x < x_1^*$, $b_1(X)$ is influenced by $\mathbb{P}(Z \neq 1|X = x)$. For the other groups, the function $b(X)$ can be analysed analogously.

Collecting all the components of $b(X)$ and after some algebra, it can be shown that

$$b(X) = \sum_{z=1}^{2} \beta_z p_z(X)[f_z(X)p_0(X) + (f_z(X) - f_j(X))p_j(X)], \tag{2.5}$$

where $z \neq j$, and $f_z(X) = \mathbb{E}(Y|X, Z = z) - \mathbb{E}(Y|X, Z = 0)$. Then, $b(X)$ is a function that depends not only on the differences between the predictions of Y with respect to a reference group, but also on the probability of being in the groups which in turn change across X.

In summary, the global marginal effect is not only a weighted average of the marginal effects in each group, but it also considers a term accounting for the differences between the predicted outcome weighted by the marginal effect of the probability of belonging to the group z. Moreover, it is not a fixed value as it changes as a function of X. In other words, the global marginal effect does not reduce to a slope, but it also considers the relevance of the predicted outcomes.

2.3 Application

The university admission system in Chile includes two mandatory selection tests (Mathematics and Language and Communication) and two elective ones (Sciences and History, Geography and Social Sciences). Other selection factors, namely, the Ranking and High school GPA are also considered in the selection process. A score, in the 150-850 scale, is assigned to each selection factor, which are weighted to obtain a unique application score.

To illustrate the interpretation of the Global Marginal Effect, we analyse the effect of the Mathematics selection test score, X, over the GPA[1] in the first year, Y, of selected students in the Faculty of Biological Sciences of a Chilean university. We analyse the three undergraduate programs offered by this Faculty: Marine Biology, Biochemistry, and Biology. The last enrolled student in each program scored 631, 631, and 623 in the Mathematics test, respectively.

[1]The scale score for the GPA is 1.0-7.0. The minimum score to pass a course is 4.0.

z	**Program**	**Prop**	γ_z
0	Marine Biology	0.20	0.0096
1	Biochemistry	0.33	0.0072
2	Biology	0.47	0.0074

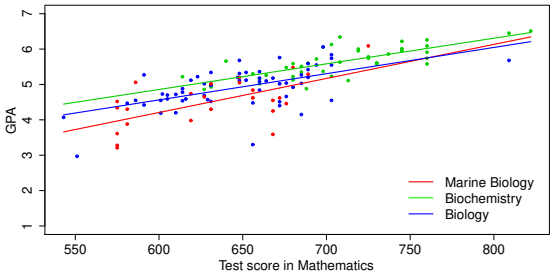
Table (2.1) γ_z and the empirical proportion of students in undergraduate programs in the faculty of Biological Sciences.

To estimate the global marginal effect in equation (2.1), the same functions described in Section 2.2.2 (i.e., a linear function $\mathbb{E}(Y|X, Z = z) = \delta_z + \gamma_z X$, and a multinomial logistic model for $p_z(X)$) were considered. By using the invariant property of the global marginal effect, the chosen reference group was the Marine Biology program.

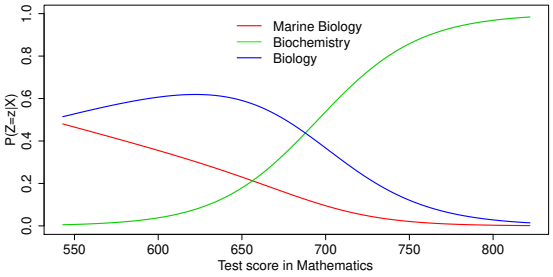
2.3.1 Results

To have a general picture on how the global marginal effects varies in terms of test scores, study programs, and the proportion of students in each program, we used the functions $a(X)$ and $b(X)$ described in the preceding section. Figure 2.2 shows a graphical representation of both functions which will be analysed together with the information provided in Table 2.1 that include the estimation of the marginal effects and the proportion of students in each program.

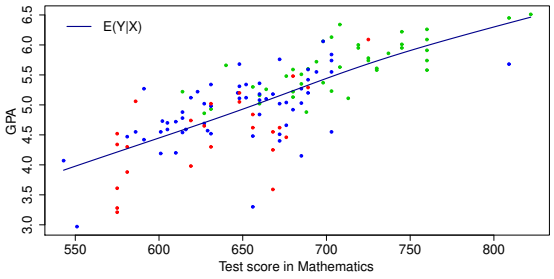
From Table 2.1, it can be seen that the Marine Biology program has the largest marginal effect and the lowest proportion of students enrolled. In contrast, the smallest marginal effect is found for the group of students enrolled in the Biochemistry program. Figure 2.2b shows the probability of belonging to each program as a function of the Mathematics test score. From the figure, it can be seen that higher score values are associated with higher probabilities of being in the Biochemistry program (i.e., $p_1(X)$ is an increasing function of X). In contrast, $p_0(X)$ (the probability of being in the Marine Biology program), is a decreasing function for all the range of scores. Regarding the Biology program group, it can be seen that up to a score 619 (approx.), $p_2(X)$ is an increasing function of X that decreases for higher score values. In summary, low scores in the Mathematics test are associated with a higher probability to find students in the Marine Biology or Biology Programs than students in Biochemistry. Likewise, higher scores in the Mathematics test are associated with a higher probability to find students in the Biochemistry than students from Marine



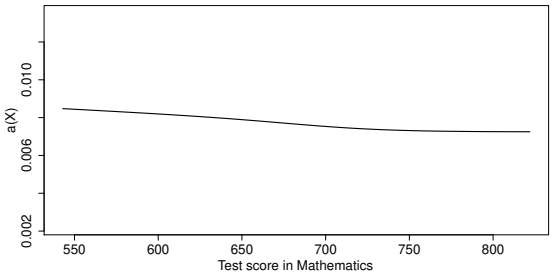
(a) $E(Y|X, Z = z) = \delta_z + \gamma_z X$



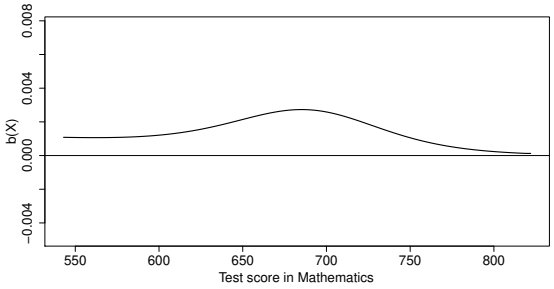
(b) $p_z(X) = F(u_z)$ and $p_0(X) = 1 - \sum_{z=1}^{2} p_z(X)$



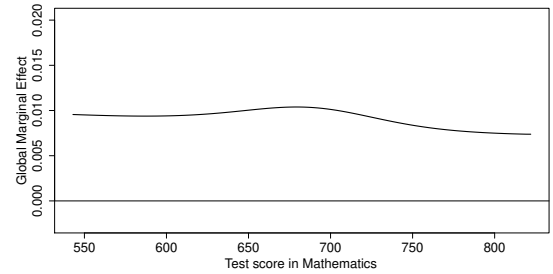
(c) Conditional Expectation using the Law of Total Probability



(d) Plot of $a(X)$



(e) Plot of $b(X)$



(f) Global Marginal Effect

Figure (2.2) Functions involved in the Global Marginal Effect

Biology or Biology programs.

The preceding analysis is useful to inspect more deeply how the two functions $a(X)$ and $b(X)$ looks like. As a matter of fact, $a(X) = \gamma_0 p_0(X) + \gamma_1 p_1(X) + \gamma_2 p_2(X)$ and thus, both γ_0 and

γ_2 have larger weights for lower values of X, while γ_1 has a larger weight for higher values of X. Considering that $\gamma_0 > \gamma_2 > \gamma_1$, it follows that $a(X)$ is a decreasing function of X as it can be seen in Figure 2.2d.

Let us analyse $b(X)$ by taking into account Equation (2.5), which can be rewritten as follows:

$$\begin{aligned} b(X) &= p_0(X)(\beta_1 p_1(X) f_1(X) + \beta_2 p_2(X) f_2(X)) \\ &+ (\beta_1 - \beta_2)(f_1(X) - f_2(X)) p_1(X) p_2(X). \end{aligned}$$

Because for low scores in the Mathematics test, $p_1(X) \to 0$, then

$$b(X) \to \beta_2 p_0(X) p_2(X) f_2(X).$$

On the other hand, for higher values of X, $p_0(X) \to 0$, $p_2(X) \to 0$, then

$$b(X) \to 0.$$

Moreover, as it is seen from Figure 2.2a, the score range $619 < x < 703$ contains most of the students from all the programs, and thus $b(X)$ is influenced by $p_0(X)$, $p_1(X)$, and $p_2(X)$, which is reflected in Figure 2.2e.

The global marginal effect, reported in Figure 2.2f, is the effect that the Mathematics test score has in students of the Faculty of Biology. For the analysed data, it turns to be positive for the whole range of test scores. For lower score values, there is a larger proportion of students belonging to a program where the marginal effect of Mathematics test score is high. In contrast, for higher scores, a larger proportion of students will be found for a program where the marginal effect is low. Note that the concavity of the curve in central range of scores is due to the fact that of both $f_1(X) > 0$ and $f_2(X) > 0$ (see Figure 2.2a).

2.4 Conclusions and Discussion

We have introduced the concept of global marginal effect which is obtained by computing the marginal effect of X on Y by decomposing $\mathbb{E}(Y|X)$ with respect to Z through the Law of Total Probability. The global marginal effect is useful when the main interest is to learn about the effect of X on Y in a partitioned population.

By means of a physiognomy of the studied population based on Figure 2.2, we have proposed a new way to analyse and interpret a marginal effect for the case of partitioned populations. Such interpretation shows the effect of X by considering other characteristics of the population (differences in predicted outcomes and the size of each group) which are accordingly defined as a non-constant function of X. Note that, although the physiognomy of the studied population considered a particular reference group, the derived result related to the invariant property of the global marginal effect with respect to the chosen reference group ensures that the type of interpretation proposed generalises no matter the group chosen as reference.

The studied scenario makes sense if both X and Y are fully observed. In the selection context, however, there is a partial observability of the outcome, whereas the explanatory random variable is fully observed (e.g., the GPA in the university is observed in selected students only, whereas the test scores are observed for all the applicants). In Chapter 3 we combine the results of this Chapter and the ideas in Chapter 1 in order to have a whole overview of the effect that a selection test score has over the GPA in the higher education system.

Chapter 3

On the marginal effect under partially observed partitioned populations: a functional predictive validity coefficient

3.1 Introduction

The impact that an exogenous explanatory random variable, X, has on the outcome random variable, Y is usually measured through the marginal effect, which quantifies the changes in the conditional expectation with respect to changes in the values of X. Several studies have attempted to measure this impact in a population that is partitioned in groups (sex, countries, age range, among others). For instance, based on quarantine measures Vasiljeva et al. (2020) conducted a regression model to assess the impact of the coronavirus pandemic on the economy of some European countries (Russia, Ukraine, Belarus, Bulgaria, Hungary, Moldova, Poland, Slovakia, Czech Republic). The analysis is made by country and the impact is reported for the whole Eastern Europe. In fact, it is concluded that

> *...GDP in Eastern Europe should be expected to decrease by 6.1% on average as a result of the COVID-19 pandemic. The main reason for this decline is the decline in production, which is crucial for all countries except Slovakia..*

The necessity to measure the impact of an exogenous random variable, over an outcome of interest

is a goal that is not far to the one in predictive validity studies, where the impact that the test scores, X, have on a random variable of interest, Y is commonly characterised for populations that are partitioned in groups (universities, programs, occupation, among others). To give an example, Grobelny (2018) studied, across fifteen occupational groups, the predictive validity of both specific and general mental ability tests over the job performance. The validity coefficient was analysed for the whole population (as if only one group existed) as well as for each occupation, where the results differ substantially with respect to the ones when only the *big* group is considered. Ayers and Peters (1977) examined the predictive validity of the Test of English as a Foreign Language, TOEFL, on the Graduate Record Examination. The subjects of study were students from Republic of China, India, Thailand, among others. According to the authors, the study does not reveal systematic differences by country. Thus, the results are reported for the total sample.

In all of these studies, the analysis is separately made for each group while the interest is to report and draw conclusions based on a global analysis. Formally, researchers are learning about the marginal effect in partitioned populations using the conditional expectation of Y conditioned not only on X, but also on a random variable, Z, characterising the groups. The most common approach is to use a multiple linear regression model with interaction terms between X and the group variable Z. By taking the difference between the marginal effect of a group of interest and one of reference, researchers compare the impact of X on Y between groups. As a matter of fact, the effect that X has on the group z with respect to the reference, z', corresponds to the "interaction effect", which is quantified by the corresponding interaction regression coefficient (see Cameron and Trivedi, 2005; Cornelissen, 2005; Powers and Xie, 1999; Norton et al., 2004; Ai and Norton, 2003; Long and Mustillo, 2018). Nevertheless, the interest is to draw global conclusions about the effect of X over Y. In this context, researchers draw conclusions by group and then for the whole population of interest. However, a trend can appear when different groups are analysed separately, and possibly disappear when they are combined. This phenomenon is related to the Simpson's Paradox (see Simpson, 1951; Blyth, 1972). An alternative analysis for learning about the marginal effect in partitioned populations is to use the Global Marginal Effect, which is introduced by Alarcón-Bustamante et al. (2021). The authors use the Law of Total Probability to combine the involved groups and draw conclusions about the predictive validity of a selection test over the GPA of enrolled students in the Faculty of Biological Sciences of a Chilean university. The groups are conformed by the three undergraduate programs offered by this Faculty: Marine

Biology, Biochemistry, and Biology.

If both X and Y were fully observed the above mentioned studies make sense; however, an additional issue arises when conducting predictive validity of selection tests studies: the outcome measured is partially observed (i.e., it is observed only in the enrolled group), whereas the scores of the selection test are observed for the whole population of applicants. This problem is accordingly called *selection problem* and arises when the sampling process does not fully reveal the behaviour of the outcome on the support of the predictors (Manski, 1993). A common practice to tackle this problem is to use available information about both test sores and the outcome of interest. As an example, Geiser and Studley (2002) conducted a predictive validity study of the SAT scores on the UC freshman grades for different socioeconomic factors. The authors declare that:

> *The only exclusions from the sample were students with missing SAT scores or high-school GPAs; students who did not complete their freshman year and/or did not have a freshman GPA recorded in the UC Corporate Student Database...*

Behind of this analysis, it is assumed a prior knowledge for the performance of the whole population. Moreover, it is assumed that the performance in the non-observed group is equal to the one in the observed groups. This assumption allows making inferences on the conditional distribution of the outcomes given the scores. However, we argue that this assumption is not pertinent because the consequence of the partial observability is that the conditional distribution of the outcome given the scores is not identified and therefore assuming any structure for it could not be assessed empirically (Manski, 1993). As the conditional distribution is not identified, the conditional expectation is not identified either. In consequence, the marginal effect is not identified. Based on the partial identification approach, Alarcón-Bustamante et al. (2020) proposed identification bounds to characterise the set of all admissible values of the marginal effect under partial observability of the outcome for the analysis of only one undergraduate program. The bounds were built assuming no specific structure for the non-observed group, as they only use desired properties of the selection tests. More specifically, it was assumed that the selection test is such that higher scores would translate to higher values of the outcome.

Using the partial identification approach, in this chapter we extend the results of Alarcón-Bustamante et al. (2021) to the partial observability of the outcome case. In fact, we conduct our study with the full available information: the test score for enrolled and non-enorolled applicants, and the GPA

for the enrolled ones.

The identification bounds for both the conditional expectation and its derivative are rigorously operationalized through the following assumptions: fisrtly, we use the above-mentioned desired property of the selection test, i.e., it is considered that there is a positive relationship between test scores and the outcome. Secondly, we use the objective that the Chilean Unique Admission System (SUA, for their initials in Spanish) has: select the best applicants according to the obtained scores in the selection tests (Centro de Estudios, MINEDUC, 2019). The implicit assumption that the SUA makes is that the selection factors are the unique ones that can predict the performance in the University. In this context, we found identification bounds under two possible scenarios, namely *The System selects correctly* and *The System selects wrongly*, which were reasoning as follows:

- if the System selects correctly, there are no more factors influencing the predictability of the outcome. As the SUA assume that the best applicants will be the best in the university, according to each selection factor, we assume that the performance of non-enrolled applicants is at most equal to the one in enrolled students.
- If the System selects wrongly, there are more factors influencing the predictability of the outcome. Thus, we assume that if only information about the selection test scores is considered, then the performance of non-enrolled applicants is at least equal to the one in enrolled students.

Both the desired property of selection tests and the possible scenarios allow us to make identification restrictions for both the conditional expectation and its derivative.

Before providing details of the proposed methodology, in the next section we briefly introduce the motivating case-study and data.

3.1.1 University admission tests in Chile

The evolution of the university admission system in Chile includes four selection tests. The *baccalaureate* test, administrated during the $1931 - 1966$ period; the *Prueba de Aptitud Académica* (PAA, for their initials in Spanish), administered during the 1967 - 2002 period; The *Prueba de Selección Universitaria* (PSU, for their initials in Spanish), administrated during the 2004-2019

period. Finally, since 2020 the *Prueba de Trancisión Universitaria* (PTU, for their initials in Spanish) is administrated. One of the main reasons for the evolution of the Chilean university admission system is the necessity to increase predictive capacity of the selection tests (Donoso, 1998).

We focus our study on the predictive validity of the PSU, which was elaborated based on the secondary school curriculum. The test includes two mandatory tests (Mathematics and Language and communication), and two elective ones (Sciences and History, Geography and Social Sciences). Additional to these tests there are other selection factors that are considered in the selection process, namely Ranking and High school Grade Point Average (HGPA). A score, in the 150-850 scale, is assigned to each selection factor, which are weighted to obtain a unique application score.

In Chile a student can apply to more than one undergraduate program with his/her test score; however it can be enrolled in one only. Based on this fact, we induced a partition of the applicants space as well as is described in Section 3.1.2.

3.1.2 Data description

The data that are used will be based on the PSU, Ranking and HGPA scores for the applicants to the Faculty of Biological Sciences of a Chilean university, X, and the GPA[1] in the first year, Y, of enrolled students. We analyse the three undergraduate programs offered by this Faculty: Marine Biology (MB), Biology (B), and Biochemistry (BC). Table (3.1) provides information about how many students applied to each program, and how many there are enrolled in each one. In this context, 58.2% of the applicants are not enrolled (NE) in this faculty (which can be considered as missing values). About the population of enrolled students, 20% there are enrolled in MB, 47.2% in B, and 32.8% in BC. Table (3.2) provides descriptive statistics for the score in each considered selection factor[2] into each enrolment status. Since the means and medians are very similar for each variable, for each group, the empirical distributions are fairly symmetric. This fact is reflected in Figure (3.1), where it can be appreciated that in all the selection factors, there is a tendency to increase the mean of the scores, such that students enrolled in MB have the minimum mean in each selection factor, and students enrolled in BC have the maximum one. In Figure (3.2) the boxplot of the GPA of observed students are shown. The tendency to increase the mean of the scores, such

[1]The scale score for the GPA is 1.0-7.0. The minimum score to pass a course is 4.0.

[2]We did not considered information about the History, Geography and Social Sciences test because we have had an additional problem related with missing data in the covariates.

Table (3.1) Application and enrolment status

		Applied to						
		MB	**B**	**BC**	**MB & B**	**MB & BC**	**B & BC**	**MB & B & BC**
Enrolled in	**NE**	18	41	107	3	0	4	1
	MB	21	0	0	2	2	0	0
	B	0	52	0	0	0	7	0
	BC	0	0	41	0	0	0	0

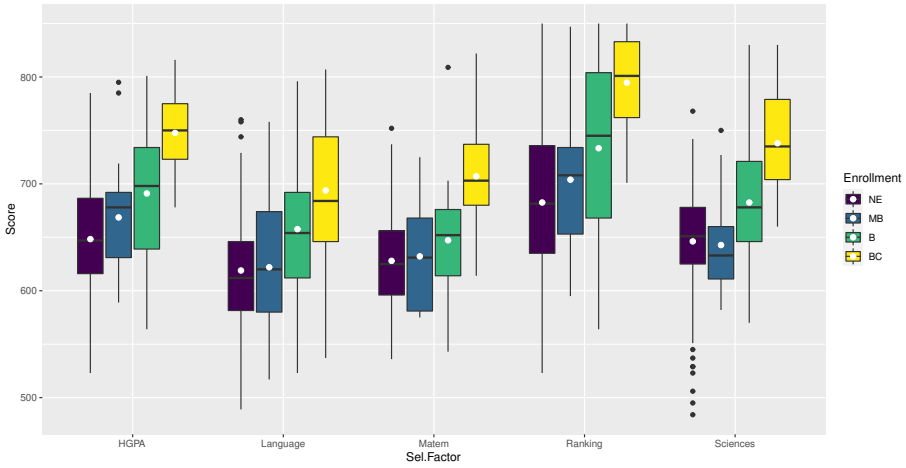


Figure (3.1) Boxplots of individual Selection factor score

that students enrolled in MB have the minimum mean GPA, and students enrolled in BC have the maximum one. This tendency is the same than the scores in each selection factor.

Table (3.2) Minimum, maximum, mean, standard deviation, and median of selection factor scores for enrolment status of the student.

Selection factor	Enrolled in	n	min	max	$\overline{x}$	SD	$Median$
Mathematics	**NE**	174	536	752	627.98	42.74	625
	MB	25	575	725	632.16	44.36	631
	B	59	543	809	647.24	44.43	652
	BC	41	614	822	707.1	45.9	703
Language	**NE**	174	489	760	618.94	51.19	612
	MB	25	517	758	621.96	62.04	620
	B	59	523	796	657.61	60.99	654
	BC	41	537	807	693.83	67.95	684
Sciences	**NE**	174	484	768	646.18	47.57	651
	MB	25	582	750	642.76	42.23	633
	B	59	570	830	682.53	55.64	678
	BC	41	660	830	738	46.69	735
HGPA	**NE**	174	523	785	648.32	53.07	647
	MB	25	589	795	668.6	51	678
	B	59	564	801	691.02	57.86	698
	BC	41	678	816	747.63	36	750
Ranking	**NE**	174	523	850	682.56	73.35	681.5
	MB	25	595	847	703.92	61.73	708
	B	59	564	850	733.34	80.09	745
	BC	41	701	850	794.68	43.93	801

3.1.3 Characterisation of the population in study

Let us define a random variable Z, where $Z = 1$ if the student applied to MB only; $Z = 2$ if the student applied to B only; $Z = 3$ if the student applied to BC only; $Z = 4$ if the student applied to the both MB and B; $Z = 5$ if the student applied to the both MB and BC; $Z = 6$ if the student applied to the both B and BC, and $Z = 7$ if the student applied to all the undergraduate programs. Let S be a random variable that represents the student's enrolment status, such that $S = 0$ if the student is not enrolled; $S = 1$ if the student is enrolled in MB; $S = 2$ if the student is enrolled in B, and $S = 3$ if the student is enrolled in BC. Note that if a student applied to MB only (i.e., $Z = 1$), it can not be enrolled in B or BC. Formally, $\{\{Z = 1\} \cap \{S = 2\}\} = \emptyset$, and $\{\{Z = 1\} \cap \{S = 3\}\} = \emptyset$. Analogously, a student that applied to both MB and B (i.e., $Z = 4$), it can not be enrolled in program BC, i.e., $\{\{Z = 4\} \cap \{S = 3\}\} = \emptyset$. In function of the two principal characteristics of the selection process: the application and the enrolment status, we have

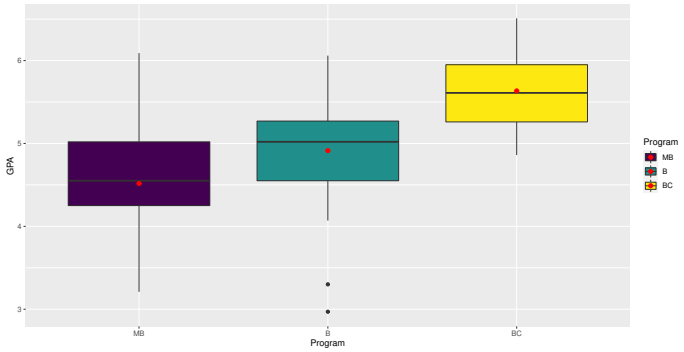


Figure (3.2) Boxplots of the GPA by undergraduate program.

established a partition of the population. In fact,

$$\sum_{i=1}^{7}\sum_{j=0}^{3}\mathbb{P}(\{Z=i\}\cap\{S=j\})=1,$$

3.1.4 Organisation of the chapter

The rest of the chapter is organised as follows. In Section 3.2 we introduce the identification bounds of the conditional expectation in the contexts of the data. Bounds are opertionalised in both scenarios: the system selects correctly and the system selects wrongly. In Section 3.3 the identification bounds for the marginal effect are shown in the context of the data. Analogously to the construction of the bounds for the conditional expectation, we use identification restrictions that are related with the two possible scenarios for the selection process and the desired property of the selection tests. Section 3.4 contains the results of the two previous sections from the Chilean university selection test case study illustrating the benefits of the proposal. The chapter finalises in Section 3.5 with conclusions and discussion.

3.2 Identification bounds for the Conditional expectation

Let Y be a random variable that represents the GPA. Let $X \in \mathcal{X}$ be the random variable that represents the score in a selection factor. Let $\mathcal{X}_{ij}$ be the space of X in the group $S=j, Z=i$. Thus, each member of the population is characterised by (Y, X, Z, S). The interest is to learn about

$\mathbb{E}(Y|X)$ in this population. By the Law of Total Probability (Kolmogorov, 1950) we have that

$$\begin{aligned}\mathbb{E}(Y|X) &= \sum_{i=1}^{7}\sum_{j=0}^{3}\mathbb{E}(Y|X,Z=i,S=j)\mathbb{P}(S=j,Z=i|X)\\ &= \sum_{i=1}^{7}\mathbb{E}(Y|X,Z=i,S=0)\mathbb{P}(S=0,Z=i|X)\\ &\quad+\sum_{i=1}^{7}\sum_{j=1}^{3}\mathbb{E}(Y|X,Z=i,S=j)\mathbb{P}(S=j,Z=i|X) \qquad (3.1)\end{aligned}$$

Note that $\mathbb{P}(S=0,Z=i|X)$ is the probability of not being observed in each group. If all the applicants had enrolled in all $i \in \{1,\ldots,7\}$, then this probability is zero, and $\mathbb{E}(Y|X)$ is equal to the weighted observed regression, which is given by the second term in the right-side of (3.1). However, in the selection problem the data generating process is uninformative about the GPA of non-enrolled students, and therefore $\mathbb{E}(Y|X,Z=i,S=0)$ is impossible to estimate. Provided that $\mathbb{P}(S=0,Z=i|X) \neq 0$, the conditional expectation of the population, $\mathbb{E}(Y|X)$ is not identified.

Several statistical procedures have been used to deal with this problem. One solution is to assume that Y is conditionally orthogonal to S in each $Z=i$, i.e, $Y \perp S|X,\{Z=i\}$[3] (see Florens and Mouchart, 1982). This fact allows making inferences on $\mathbb{E}(Y|X)$ *ignoring* the non-observed values of Y, which can lead to underestimation of the predictive capacity of the selection test (Alarcón-Bustamante et al., 2020). Other solution is based on regression models with truncated distributions (for instance, see Nawata, 1994; Heckman, 1976, 1979; Marchenko and Genton, 2012), where it is explicitly assumed that the conditional distribution of the outcome given the scores is known up to some parameters. Note that in both cases $\mathbb{E}(Y|X,S=0,Z=i)$ is assumed known, but the data generation process does not reveal information about it, and therefore the conclusions can be seriously affected.

According to the ideas of Manski (1989, 2003, 2007, 1993), we can define an identification region for $\mathbb{E}(Y|X)$ when some weaker assumptions about the conditional expectation in the non-observed group are made. Proposition 3.2.1 give us the identification region for $\mathbb{E}(Y|X=x)$ when it is assumed that $\mathbb{E}(Y|X,S=0,Z=i)$ is bounded:

[3] It is also known as ignorability assumption.

Proposition 3.2.1. *If* $\mathbb{P}(Y \in [y_{0ix}, y_{1ix}]|X = x, Z = i, S = 0) = 1$, *then* $\mathbb{E}(Y|X = x, Z = i, S = 0) \in [y_{0ix}, y_{1ix}]$, *where* $y_{0ix} \leq y_{1ix}$. *Thus, the identification region for* $\mathbb{E}(Y|X = x)$ *is given by:*

$$\mathbb{E}(Y|X=x) \in \Bigg[\mathbb{E}(Y_{0x}|X=x,S=0)\mathbb{P}(S=0|X=x)+\sum_{i=1}^{7}\sum_{j=1}^{3}E_{ji}(x)p_{ji}(x); \quad \mathbb{E}(Y_{1x}|X=x,S=0)\mathbb{P}(S=0|X=x)+\sum_{i=1}^{7}\sum_{j=1}^{3}E_{ji}(x)p_{ji}(x)\Bigg] \tag{3.2}$$

where $p_{ji}(x) = \mathbb{P}(S = j, Z = i|X = x)$, $E_{ji}(x) = \mathbb{E}(Y|X = x, Z = i, S = j)$, $Y_{0x} = (y_{01x}, \ldots, y_{07x})$, *and* $Y_{1x} = (y_{11x}, \ldots, y_{17x})$

(see a proof in Appendix B).

The common term in this identification region is $\mathbb{E}(Y|X = x, S \neq 0) = \sum_{i=1}^{7}\sum_{j=1}^{3} E_{ji}(x)p_{ji}(x)$, the weighted observed regression. If all applicants had enrolled, $\mathbb{P}(S = 0|X = x) = 0$, thus $\mathbb{E}(Y|X)$ is point identified. Nevertheless, in the selection problem $\mathbb{P}(S = 0|X = x) > 0$, then $\mathbb{E}(Y|X)$ is partially identified (Manski, 2003). Both $\mathbb{E}(Y_{0x}|S = 0, X = x)$ and $\mathbb{E}(Y_{1x}|S = 0, X = x)$ are impossible to estimate because applicants in $S = 0$ there are not enrolled and therefore its GPA is not observed. Note that y_{0ix} corresponds to the minimum GPA we are ready to believe for non-enrolled students in each group $Z = i$, which is a function that depends on the score of enrolled students (analogously, y_{1ix} corresponds to the maximum one). Thus, $\mathbb{E}(Y_{0x}|S = 0, X = x)$ is the weighted average of the minimum GPA that we are ready to believe in each group, and each score $X = x$, which is weighted by the probability of not being enrolled with the correspondent score $X = x$, $\mathbb{P}(S = 0|X = x)$ (analogously for $\mathbb{E}(Y_{1x}|S = 0, X = x)$). Moreover, the width of the region is $\mathbb{E}(Y_{1x} - Y_{0x}|S = 0, X = x)\mathbb{P}(S = 0|X = x)$. Thus the sternness of the identification problem depends directly on $\mathbb{P}(S = 0|X = x)$, the probability of not being observed (Manski, 2003).

Remark 3.2.1. *The wider theoretical identification bounds are achieved when* $y_{0ix} = y_0$, *and* $y_{1ix} = y_1$ *(i.e., the minimum and the maximum theoretical GPA, respectively).*

The theoretical identification bounds are defined by the minimum and the maximum possible GPA. By considering that the scale score for the GPA in Chile is 1.0-7.0, we have that $y_{0ix} = 1.0$ and

$y_{1ix} = 7.0$. In this case, the lower bound of the conditional expectation is interpreted as the value $\mathbb{E}(Y|X)$ takes if, in the non-enrolled applicants, Y is always equal to $y_0 = 1.0$ (i.e., if all students obtained the worst GPA). Regarding the upper bound, it is interpreted as the value $\mathbb{E}(Y|X)$ takes if, the non-enrolled applicants, Y is always equal to $y_1 = 7.0$ (i.e., if all students obtained the best GPA) (Manski, 1989).

Remark 3.2.2. *The wider empirical identification bounds are achieved when $y_{0ix} = m_{0i}$ and $y_{1ix} = m_{1i}$, the minimum and the maximum observed GPA into the group $Z = i$, respectivelly .*

The empirical identification bounds are defined by the minimum and the maximum observed GPA in each group $Z = i$. The lower bound of the conditional expectation is interpreted as the value $\mathbb{E}(Y|X)$ takes if, for each $Z = i, S = 0$, the GPA, Y, is always equal to the minimum observed one in $Z = i, S \neq 0$. The upper bound is interpreted as the value $\mathbb{E}(Y|X = x)$ takes if, for each $Z = i, S = 0$, the GPA, Y, is always equal to the maximum observed one in $Z = i, S \neq 0$.

In order to find identification restrictions for $\mathbb{E}(Y|X, Z = i, S = 0)$, we write it in terms of the conditional expectation of the enrolled ones in the same group, i.e., $\mathbb{E}(Y|X, Z = i, S = 0)$ is written in terms of $\mathbb{E}(Y|X, Z = i, S \neq 0)$ as follows:

- $\mathbb{E}(Y|X, Z = i, S = 0)$ can be written in terms of $\mathbb{E}(Y|X, Z = i, S = i)$, for all $i \in \{1, 2, 3\}$.
- $\mathbb{E}(Y|X, Z = 4, S = 0)$ can be written in terms of both $\mathbb{E}(Y|X, Z = 4, S = 1)$ and $\mathbb{E}(Y|X, Z = 4, S = 2)$ only.
- $\mathbb{E}(Y|X, Z = 5, S = 0)$ can be written in terms of both $\mathbb{E}(Y|X, Z = 5, S = 1)$ and $\mathbb{E}(Y|X, Z = 5, S = 3)$ only.
- $\mathbb{E}(Y|X, Z = 6, S = 0)$ can be written in terms of both $\mathbb{E}(Y|X, Z = 6, S = 2)$ and $\mathbb{E}(Y|X, Z = 6, S = 3)$ only.
- $\mathbb{E}(Y|X, Z = 7, S = 0)$ can be written in terms of $\mathbb{E}(Y|X, Z = 7, S = 1)$, $\mathbb{E}(Y|X, Z = 7, S = 2)$, and $\mathbb{E}(Y|X, Z = 7, S = 3)$.

From the empirical point of view, a score in the selection factor, X, can be observed in the group of non-enrolled applicants even if it is not observed in the group of enrolled students. By taking into account this issue, in the next sections we define the corresponding y_{0ix} and y_{1ix} in both scenarios: *The system selects correctly*, and *The system selects wrongly*.

Scenario 1: The system selects correctly

If the system selects correctly, we can assume that the performance of non-enrolled applicants would not be better than the enrolled students. In terms of the conditional expectation, we assume that, for each $Z = i$, the not observed expected GPA is at most the the one of an enrolled student with the same score, X. In fact, for $i \in \{1, 2, 3\}$ we assume that $\mathbb{E}(Y|X = x, Z = i, S = 0) \leq \mathbb{E}(Y|X = x, Z = i, S = i)$. When $i \geq 3$, we can write $\mathbb{E}(Y|X = i, Z = i, S = 0)$ in function of more than one observed conditional expectation. Note that, this identification restriction makes more informative the upper bound, such that we are assuming that

$$y_{0ix} \leq \mathbb{E}(Y|X = x, Z = i, S = 0) \leq \mathbb{E}(Y|X = x, Z = i, S \neq 0).$$

Thus, we define y_{1ix} for all the groups $Z = i$ as follows: let $\mathbb{1}_{\{\mathcal{X}_{ji}\}}$ be a binary random variable, such that $\mathbb{1}_{\{\mathcal{X}_{ji}\}} = 1$ if $x \in \mathcal{X}_{ji}$ and 0 otherwise. Thus, for each $X = x$, we have that

$$y_{1ix} := \begin{cases} \mathbb{E}(Y|X = x, Z = i, S = i) \cdot \mathbb{1}_{\{\mathcal{X}_{ii}\}} + m_{1i} \cdot (1 - \mathbb{1}_{\{\mathcal{X}_{ii}\}}) & \text{if } i \in \{1, 2, 3\} \\ \min\limits_{x \in \{\mathcal{X}_{14} \cup \mathcal{X}_{24}\}} [E_{14}(x), E_{24}(x)] \cdot \mathbb{1}_{\{\mathcal{X}_{14} \cup \mathcal{X}_{24}\}} + m_{14} \cdot (1 - \mathbb{1}_{\{\mathcal{X}_{14} \cup \mathcal{X}_{24}\}}) & \text{if } i = 4 \\ \min\limits_{x \in \{\mathcal{X}_{15} \cup \mathcal{X}_{35}\}} [E_{15}(x), E_{35}(x)] \cdot \mathbb{1}_{\{\mathcal{X}_{15} \cup \mathcal{X}_{35}\}} + m_{15} \cdot (1 - \mathbb{1}_{\{\mathcal{X}_{15} \cup \mathcal{X}_{35}\}}) & \text{if } i = 5 \\ \min\limits_{x \in \{\mathcal{X}_{26} \cup \mathcal{X}_{36}\}} [E_{26}(x), E_{36}(x)] \cdot \mathbb{1}_{\{\mathcal{X}_{26} \cup \mathcal{X}_{36}\}} + m_{16} \cdot (1 - \mathbb{1}_{\{\mathcal{X}_{26} \cup \mathcal{X}_{36}\}}) & \text{if } i = 6 \\ \min\limits_{x \in \{\mathcal{X}_{17} \cup \mathcal{X}_{27} \cup \mathcal{X}_{37}\}} [E_{17}(x), E_{27}(x), E_{37}(x)] \cdot \mathbb{1}_{\{\mathcal{X}_{17} \cup \mathcal{X}_{27} \cup \mathcal{X}_{37}\}} & \\ \qquad + m_{17} \cdot (1 - \mathbb{1}_{\{\mathcal{X}_{17} \cup \mathcal{X}_{27} \cup \mathcal{X}_{37}\}}) & \text{if } i = 7 \end{cases}$$

For each case, we are assuming that if no-enrolled applicant can not be compared with any enrolled one with the same score $X = x$, then its expected GPA can not be higher than the maximum observed one in each group. i.e., $y_{1ix} = m_{1i}$. Regarding y_{0ix}, we assume that the GPA of a non-enrolled takes the minimum observed GPA in each group, i.e., $y_{0ix} = m_{0i}$.

Under these identification restrictions and taking into account Proposal 3.2.1, we have that when it is assumed that the System chooses correctly:

$$\mathbb{E}(Y_{0x}|X = x, S = 0) = \sum_{i=1}^{7} m_{0i} \mathbb{P}(Z = i|X = x, S = 0),$$

and

$$
\begin{aligned}
\mathbb{E}(Y_{1x}|X=x,S=0) &= \sum_{i=1}^{3}\left[E_{ii}(x)\cdot\mathbb{1}_{\{\mathcal{X}_{ii}\}}+m_{1i}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{ii}\}})\right]\mathbb{P}(Z=i|X=x,S=0)\\
&+ \left[\min_{x\in\{\mathcal{X}_{14}\cup\mathcal{X}_{24}\}}[E_{14}(x),E_{24}(x)]\cdot\mathbb{1}_{\{\mathcal{X}_{14}\cup\mathcal{X}_{24}\}}+m_{14}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{14}\cup\mathcal{X}_{24}\}})\right]\\
&\quad\times\mathbb{P}(Z=4|X=x,S=0)\\
&+ \left[\min_{x\in\{\mathcal{X}_{15}\cup\mathcal{X}_{35}\}}[E_{15}(x),E_{35}(x)]\cdot\mathbb{1}_{\{\mathcal{X}_{15}\cup\mathcal{X}_{35}\}}+m_{15}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{15}\cup\mathcal{X}_{35}\}})\right]\\
&\quad\times\mathbb{P}(Z=5|X=x,S=0)\\
&+ \left[\min_{x\in\{\mathcal{X}_{26}\cup\mathcal{X}_{36}\}}[E_{26}(x),E_{36}(x)]\cdot\mathbb{1}_{\{\mathcal{X}_{26}\cup\mathcal{X}_{36}\}}+m_{16}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{26}\cup\mathcal{X}_{36}\}})\right]\\
&\quad\times\mathbb{P}(Z=6|X=x,S=0)\\
&+ \left[\min_{x\in\{\mathcal{X}_{17}\cup\mathcal{X}_{27}\cup\mathcal{X}_{37}\}}[E_{17}(x),E_{27}(x),E_{37}(x)]\cdot\mathbb{1}_{\{\mathcal{X}_{17}\cup\mathcal{X}_{27}\cup\mathcal{X}_{37}\}}\right.\\
&\qquad\left.+m_{17}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{17}\cup\mathcal{X}_{27}\cup\mathcal{X}_{37}\}})\right]\times\mathbb{P}(Z=7|X=x,S=0) \qquad (3.3)
\end{aligned}
$$

Scenario 2: The system selects wrongly

In contrast to the above-mentioned scenario, if the system selects wrongly we can assume that the performance of non-enrolled applicants will be better than the enrolled students. Translated to the expected GPA, given tests scores in each group, we assume that $\mathbb{E}(Y|X=x,Z=i,S=0)\geq\mathbb{E}(Y|X=x,Z=i,S=i)$ for $i\in\{1,2,3\}$. Analogously to the above scenario, we can write $\mathbb{E}(Y|X=x,Z=i,S=0)$ in function of more than one observed conditional expectation when $i>3$. This scenario allow us to make more informative the lower bound, as we are assuming that

$$\mathbb{E}(Y|X=x,Z=i,S\neq 0)\leq\mathbb{E}(Y|X=x,Z=i,S=0)\leq y_{1ix}.$$

Hence, we define y_{0ix} as follows:

$$y_{0ix} := \begin{cases} \mathbb{E}(Y|X=x,Z=i,S=i)\cdot \mathbb{1}_{\{\mathcal{X}_{ii}\}} + m_{0i}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{ii}\}}) & \text{if } i\in\{1,2,3\} \\ \max\limits_{x\in\{\mathcal{X}_{14}\cup\mathcal{X}_{24}\}} [E_{14}(x),E_{24}(x)]\cdot \mathbb{1}_{\{\mathcal{X}_{14}\cup\mathcal{X}_{24}\}} + m_{04}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{14}\cup\mathcal{X}_{24}\}}) & \text{if } i=4 \\ \max\limits_{x\in\{\mathcal{X}_{15}\cup\mathcal{X}_{35}\}} [E_{15}(x),E_{35}(x)]\cdot \mathbb{1}_{\{\mathcal{X}_{15}\cup\mathcal{X}_{35}\}} + m_{05}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{15}\cup\mathcal{X}_{35}\}}) & \text{if } i=5 \\ \max\limits_{x\in\{\mathcal{X}_{26}\cup\mathcal{X}_{36}\}} [E_{26}(x),E_{36}(x)]\cdot \mathbb{1}_{\{\mathcal{X}_{26}\cup\mathcal{X}_{36}\}} + m_{06}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{26}\cup\mathcal{X}_{36}\}}) & \text{if } i=6 \\ \max\limits_{x\in\{\mathcal{X}_{17}\cup\mathcal{X}_{27}\cup\mathcal{X}_{37}\}} [E_{17}(x),E_{27}(x),E_{37}(x)]\cdot \mathbb{1}_{\{\mathcal{X}_{17}\cup\mathcal{X}_{27}\cup\mathcal{X}_{37}\}} & \\ \qquad +m_{07}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{17}\cup\mathcal{X}_{27}\cup\mathcal{X}_{37}\}}) & \text{if } i=7 \end{cases}$$

For each case, we are assuming that if a no-enrolled applicant can not be compared with any enrolled one with the same score $X=x$, then its expected GPA can be higher than the minimum observed one in each group. i.e., $y_{0ix} = m_{0i}$. Regarding y_{1ix}, we assume that the GPA of a non-enrolled applicant can takes the maximum observed GPA in each group, i.e., $y_{1ix} = m_{1i}$.
Under these identification restrictions and taking into account Proposal 3.2.1, we have that when it is assumed that the System selects wrongly:

$$\begin{aligned} \mathbb{E}(Y_{0x}|X=x,S=0) &= \sum_{i=1}^{3}\left[E_{ii}(x)\cdot\mathbb{1}_{\{\mathcal{X}_{ii}\}} + m_{0i}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{ii}\}})\right]\mathbb{P}(Z=i|X=x,S=0) \\ &+ \left[\max_{x\in\{\mathcal{X}_{14}\cup\mathcal{X}_{24}\}} [E_{14}(x),E_{24}(x)]\cdot\mathbb{1}_{\{\mathcal{X}_{14}\cup\mathcal{X}_{24}\}} + m_{04}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{14}\cup\mathcal{X}_{24}\}})\right] \\ &\quad\times\mathbb{P}(Z=4|X=x,S=0) \\ &+ \left[\max_{x\in\{\mathcal{X}_{15}\cup\mathcal{X}_{35}\}} [E_{15}(x),E_{35}(x)]\cdot\mathbb{1}_{\{\mathcal{X}_{15}\cup\mathcal{X}_{35}\}} + m_{05}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{15}\cup\mathcal{X}_{35}\}})\right] \\ &\quad\times\mathbb{P}(Z=5|X=x,S=0) \\ &+ \left[\max_{x\in\{\mathcal{X}_{26}\cup\mathcal{X}_{36}\}} [E_{26}(x),E_{36}(x)]\cdot\mathbb{1}_{\{\mathcal{X}_{26}\cup\mathcal{X}_{36}\}} + m_{06}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{26}\cup\mathcal{X}_{36}\}})\right] \\ &\quad\times\mathbb{P}(Z=6|X=x,S=0) \\ &+ \left[\max_{x\in\{\mathcal{X}_{17}\cup\mathcal{X}_{27}\cup\mathcal{X}_{37}\}} [E_{17}(x),E_{27}(x),E_{37}(x)]\cdot\mathbb{1}_{\{\mathcal{X}_{17}\cup\mathcal{X}_{27}\cup\mathcal{X}_{37}\}}\right. \\ &\qquad \left.+m_{07}\cdot(1-\mathbb{1}_{\{\mathcal{X}_{17}\cup\mathcal{X}_{27}\cup\mathcal{X}_{37}\}})\right]\times\mathbb{P}(Z=7|X=x,S=0), \end{aligned} \tag{3.4}$$

and

$$\mathbb{E}(Y_{1x}|X=x,S=0) = \sum_{i=1}^{7} m_{1i}\mathbb{P}(Z=i|X=x,S=0).$$

In summary, we have proposed identification bounds for the conditional GPA in this partitioned population in two possible scenarios. The wider bounds are achieved using the range of possible

values of the GPA in both theoretical and empirical cases. The assumptions regarding the belief about the correctness the system has to select the applicants allow us to make more informative the bounds. In fact, if we are ready to believe that the System selects correctly, then the upper bound will be more informative. In contrast, if we are ready to believe that the System selects wrongly, then the lower bound will be more informative.

In the next section we provide identification bounds for the marginal effect of the selection factor scores over the GPA, which is considered as a validity coefficient for them. The bounds are operacionalized taking into account the same scenarios than for the conditional expectation.

3.3 Identification bounds for the impact of the selection test score over the GPA

The effect that the score, X, has on the outcome random variable of interest, Y, is usually measured through the marginal effect, which is accordingly defined as the derivative of the conditional expectation, $\mathbb{E}(Y|X)$, with respect to X, namely:

$$ME^X = \frac{d\mathbb{E}(Y|X)}{dX}.$$

In our context, we need to characterise the marginal effect in a population that is partitioned by the application status and the enrolment one.

We know that each $Z = i$ is partitioned by S, where $S \in \{0, 1, 2, 3\}$. According to the ideas of Alarcón-Bustamante et al. (2021), where the marginal effect for partitioned populations is defined, namely Global Marginal Effect (GME), we have that

$$\begin{aligned} ME^X = & \sum_{i=1}^{7} \left[\sum_{j=0}^{3} \frac{d\mathbb{E}(Y|X, Z=i, S=j)}{dX} \mathbb{P}(S=j|X, Z=i) \right. \\ & + \sum_{j=1}^{3} [\mathbb{E}(Y|X, Z=i, S=j) - \mathbb{E}(Y|X, Z=i, S=0)] \\ & \left. \times \frac{d\mathbb{P}(S=j|X, Z=i)}{dX} \right] \mathbb{P}(Z=i|X) \\ & + \sum_{i=1}^{7} \left[\sum_{j=0}^{3} \mathbb{E}(Y|X, Z=i, S=j) \mathbb{P}(S=j|X, Z=i) \right] \frac{d\mathbb{P}(Z=i|X)}{dX}. \end{aligned} \tag{3.5}$$

Intuitively, the marginal effect in this partitioned population is the weighted average of each GME in $Z = i$. However, from (3.5), it can be seen that an additional term, related with the expected GPA in each $Z = i$, and the marginal marginal effect observed through the categorical regression, $\mathbb{P}(Z = i|X)$, appears. In particular, it follows that if $Z \perp\!\!\!\perp X$ (i.e., if the application status of the student does not depend on the obtained score, X), equation (3.5) reduces to a weighted average of each GME (wGME). However, in the selection process this fact does not make sense because the student applies according to the obtained score, hence we can assume that $Z \not\perp\!\!\!\perp X$. For each $Z = i$, by considering the non-enrolled group as the reference one, the GME is computed, and it is weighted by the probability of belong to each $Z = i$. Note that if $S \perp\!\!\!\perp X|\{Z = i\}$ (i.e., enrolment status does not depend on the score, X in each application status $Z = i$), then each GME corresponds to the weighted average marginal effect. However, in the selection process context this fact does not make sense because for all $Z = i$, the students are enrolled (or not) according to the the obtained score. Assuming that $S \not\perp\!\!\!\perp X|\{Z = i\}$, it can be seen that the wGME depends not only on the GME in each group, but also on the differences of the predicted GPAs with respect to the one of the non-enrolled applicants. Nevertheless, both the last quantity and its derivative, are impossible to obtain because the GPA of the non-enrolled applicants is not observed. This fact relies on a non identification of ME^X. A possible solution is to use the observed information only as well as the above mentioned procedures, and compute the derivative of $\mathbb{E}(Y|X, Z = i, S \neq 0)$ through with respect to X. However, assuming any structure for the non-observed quantities can seriously affect the conclusions.

In order to use the partial identification approach, it is important to highlight that a bound on Y does not by itself restrict the marginal effect. A bound on Y combined with one on the marginal effect for non-enrolled applicants does (Manski, 1989). In this context, we combine the identification restriction in Proposition 3.2.1 with an assumption on the non-observed marginal effect. In fact, we assume that the marginal effect for the non-enrolled population exist for each $Z = i$ (i.e., the marginal effect is bounded, which means that if this population had been enrolled, the score of the selection test would have predicted the outcome with an associated marginal effect). Following the ideas of Manski (1989). Proposition 3.3.1 give us the identification region for the marginal effect:

Proposition 3.3.1. *Let us suppose that* $y_{0ix} \leq \mathbb{E}(Y|X, Z = i, S = 0) \leq y_{1ix}$, *and that the marginal effect in non-enrolled students exists, for each* $Z = i$, *i.e.,*

$$D_{0ix} \leq \frac{d\mathbb{E}(Y|X, Z = i, S = 0)}{dX} \leq D_{1ix},$$

with $D_{0ix} \leq D_{1ix}$; hence, the identification bound for the marginal effect of X over Y, evaluated at $X = x$, namely $ME^{X=x}$ is given by:

$$
\begin{aligned}
ME^{X=x} \in \Bigg[& \mathbb{E}(D_{0x}|X=x,S=0)\mathbb{P}(S=0|X=x) + E\left(f_1^+(X) - f_0^-(X)|X=x\right) \\
& + \sum_{i=1}^{7}\left[g_{0i}^+(x) - g_{1i}^-(x)\right]\mathbb{P}(S=0|X=x,Z=i) \\
& + \sum_{i=1}^{7}\left\{\sum_{j=1}^{3}\left.\frac{d\mathbb{E}(Y|X=x,S=j,Z=i)}{dX}\right|_{X=x}\mathbb{P}(S=j|X=x,Z=i)\right\}\mathbb{P}(Z=i|X=x) \\
& + \sum_{i=1}^{7}\left\{\sum_{j=1}^{3}\mathbb{E}(Y|X=x,S=j,Z=i)\mathbb{P}(S=j|X=x,Z=i)\right\}\left.\frac{d\mathbb{P}(Z=i|X)}{dX}\right|_{X=x}; \quad (3.6) \\
& \mathbb{E}(D_{1x}|X=x,S=0)\mathbb{P}(S=0|X=x) + E\left(f_0^+(X) - f_1^-(X)|X=x\right) \\
& + \sum_{i=1}^{7}\left[g_{1i}^+(x) - g_{0i}^-(x)\right]\mathbb{P}(S=0|X=x,Z=i) \\
& + \sum_{i=1}^{7}\left\{\sum_{j=1}^{3}\left.\frac{d\mathbb{E}(Y|X=x,S=j,Z=i)}{dX}\right|_{X=x}\mathbb{P}(S=j|X=x,Z=i)\right\}\mathbb{P}(Z=i|X=x) \\
& + \sum_{i=1}^{7}\left\{\sum_{j=1}^{3}\mathbb{E}(Y|X=x,S=j,Z=i)\mathbb{P}(S=j|X=x,Z=i)\right\}\left.\frac{d\mathbb{P}(Z=i|X)}{dX}\right|_{X=x}\Bigg],
\end{aligned}
$$

where

- $D_{0x} = (D_{01x}, \ldots D_{07x})$; $D_{1x} = (D_{11x}, \ldots D_{17x})$,
- $f_1^-(X) = (f_{11}^-(X), \ldots, f_{17}^-(X))$; $f_1^+(X) = (f_{11}^+(X), \ldots, f_{17}^+(X))$,
- $f_0^-(X) = (f_{01}^-(X), \ldots, f_{07}^-(X))$; $f_0^+(X) = (f_{01}^+(X), \ldots, f_{07}^+(X))$, *with*
- $f_{1i}^+(X) = \sum_{j=1}^{3}\left(\mathbb{E}(Y|X,Z=i,S=j) - y_{1ix}\right)\frac{d^+\mathbb{P}(S=j|X,Z=i)}{dX}$,
- $f_{1i}^-(X) = \sum_{j=1}^{3}\left(\mathbb{E}(Y|X,Z=i,S=j) - y_{1ix}\right)\frac{d^-\mathbb{P}(S=j|X,Z=i)}{dX}$,
- $f_{0i}^+(X) = \sum_{j=1}^{3}\left(\mathbb{E}(Y|X,Z=i,S=j) - y_{0ix}\right)\frac{d^+\mathbb{P}(S=j|X,Z=i)}{dX}$,
- $f_{0i}^-(X) = \sum_{j=1}^{3}\left(\mathbb{E}(Y|X,Z=i,S=j) - y_{0ix}\right)\frac{d^-\mathbb{P}(S=j|X,Z=i)}{dX}$,

- $g_{0i}^{+}(x) = y_{0ix} \left.\dfrac{d^{+}\mathbb{P}(Z=i|X)}{dX}\right|_{X=x}$; $g_{1i}^{+}(x) = y_{1ix} \left.\dfrac{d^{+}\mathbb{P}(Z=i|X)}{dX}\right|_{X=x}$,
- $g_{0i}^{-}(x) = y_{0ix} \left.\dfrac{d^{-}\mathbb{P}(Z=i|X)}{dX}\right|_{X=x}$; $g_{1i}^{-}(x) = y_{1ix} \left.\dfrac{d^{-}}{dX}\mathbb{P}(Z=i|X)\right|_{X=x}$, *and*
- $\dfrac{d^{+}\lambda(X)}{dX}$, *and* $\dfrac{d^{-}\lambda(X)}{dX}$ *are the positive and negative part of* $\frac{d\lambda(X)}{dX}$, *respectively.*

(see a proof in Appendix C).

The common term in 3.6 corresponds to the the sum of the weighted average of the observed weighted marginal effect; and the sum of the observed regression in $Z = i$, multiplied by the effect that X has on the probability of being to any application status. Analogously to the identification region for the conditional expectation, if all the applicants had enrolled, then the marginal effect is point identified. However, in our context the non-enrolled applicants cause missing values on the GPA, hence the marginal effect is not identified. In fact, the width, $W(x)$, of the region is:

$$\begin{aligned} W(x) \quad = \quad & \mathbb{E}(D_{1x} - D_{0x}|X=x, S=0)\mathbb{P}(S=0|X=x) \\ & + \sum_{i=1}^{7} (y_{ix} - y_{0ix}) \left[\left| \left.\frac{d\mathbb{P}(Z=i|X)}{dX}\right|_{X=x} \right| \mathbb{P}(S=0|X,Z=i) \right. \\ & \left. + \sum_{j=1}^{3} \left| \left.\frac{d\mathbb{P}(S=j|X,Z=i)}{dX}\right|_{X=x} \right| \mathbb{P}(Z=i|X=x) \right] \end{aligned} \tag{3.7}$$

(see a proof in Appendix D), which varies with respect to both the probability of missing GPAs in the whole population and the probability of missing GPAs in each $Z = i$; hence, the severity of the identification problem is directly related with the probability that an applicant, with certain score $X = x$, will not enrolled in a program. Provided that $y_{0ix} \leq y_{1ix}$, and $D_{0ix} \leq D_{1ix}$, from (3.7) it can be seen that $W(x) \geq 0$ for all $X = x$, thus we assure that the upper bound of the identification region is greater than the lower one.

Although the non-observed expected GPA can be bounded by the range of possible values of it, the non-observed derivative of the expected GPA, given the test scores, is rarely bounded (Manski, 1989). By the desired selection test property, we can assume that this derivative would be non-negative. This fact allow us to assume that the minimum marginal effect in a non-enrolled student could have is a null effect. Regarding the upper bound for the derivative of the expected GPA, it is so rare to find one; hence we propose to use the maximum observed marginal effect. This fat

relies on supposing that the predictability of X over Y in a non-observed group is at most equal to the maximum one in the whole observed group. This assumption is realistic because one objective of selection tests is to choose those applicants that would obtain a better outcome than those who were not selected. In terms of the marginal effect, these restrictions are translated as:

$$0 \leq \left.\frac{d\mathbb{E}(Y|X, Z=i, S=0)}{dX}\right|_{X=x} \leq \max_{x\in\mathcal{X}} \left\{ \left.\frac{d\mathbb{E}(Y|X, S\neq 0)}{dX}\right|_{X=x} \right\} \tag{3.8}$$

Inequality (3.8) give us the wider identification bounds, which are given by

$$\begin{aligned} ME^{X=x} \in \Bigg[& E\left(f_1^+(X) - f_0^-(X)|X=x\right) + \sum_{i=1}^{7}\left[g_{0i}^+(x) - g_{1i}^-(x)\right]\mathbb{P}(S=0|X,Z=i) \\ & + \sum_{i=1}^{7}\left\{\sum_{j=1}^{3} \left.\frac{d\mathbb{E}(Y|X=x,S=j,Z=i)}{dX}\right|_{X=x} \mathbb{P}(S=j|X=x,Z=i)\right\} \\ & \quad \times\mathbb{P}(Z=i|X=x) \\ & + \sum_{i=1}^{7}\left\{\sum_{j=1}^{3} \mathbb{E}(Y|X=x,S=j,Z=i)\mathbb{P}(S=j|X,Z=i)\right\} \frac{d\mathbb{P}(Z=i|X)}{dX}; \\ & \max_{x\in\mathcal{X}}\left\{\left.\frac{d\mathbb{E}(Y|X,S\neq 0)}{dX}\right|_{X=x}\right\}\mathbb{P}(S=0|X=x) \\ & + E\left(f_0^+(X) - f_1^-(X)|X=x\right) + \sum_{i=1}^{7}\left[g_{1i}^+(x) - g_{0i}^-(x)\right]\mathbb{P}(S=0|X,Z=i) \\ & + \sum_{i=1}^{7}\left\{\sum_{j=1}^{3} \left.\frac{d\mathbb{E}(Y|X=x,S=j,Z=i)}{dX}\right|_{X=x} \mathbb{P}(S=j|X=x,Z=i)\right\} \\ & \quad \times\mathbb{P}(Z=i|X=x) \\ & + \sum_{i=1}^{7}\left\{\sum_{j=1}^{3} \mathbb{E}(Y|X=x,S=j,Z=i)\mathbb{P}(S=j|X,Z=i)\right\} \frac{d\mathbb{P}(Z=i|X)}{dX}\Bigg]. \end{aligned} \tag{3.9}$$

To find the wider identification bounds, we need to combine assumptions about both the non-observed marginal effect and the non-observed GPAs. This fact allow us to establish the following remarks:

Remark 3.3.1. *The wider empirical-theoretical identification bounds for the marginal effect are achieved by combining both the empirical identification restriction* (3.8) *and the theoretical one* $y_{0ix} = y_0$, *and* $y_{1ix} = y_1$.

Remark 3.3.2. *The wider empirical-empirical identification bounds for the marginal effect are achieved by combining both the empirical identification restriction* (3.8) *and the empirical one* $y_{0ix} = m_{0i}$, *and* $y_{1ix} = m_{1i}$.

We have described the general identification region for the marginal effect of X over Y and the wider one. However, we need to make identification restrictions in order to obtain the bounds from the empirical evidence. Assumptions for both the marginal effect in each group and the expected GPAs in non-enrolled applicants, are accordingly chosen by the above mentioned scenarios: *The system selects correctly*, and *The system selects wrongly*. In section 3.2 we proposed identification restrictions about both y_{0ix} and y_{1ix} according to each scenario. The identification restrictions for the non-observed marginal effect are analogously defined as well as for the non-observed conditional expectation, i.e., in function of the observed ones in each partition of $Z = i$.

Scenario 1: The system selects correctly

Note that the SUA selects according to the scores in the selection test. An implicit assumption that it makes is that the test score is the unique factor that has an effect over the performance in the University, which is typically measured through the GPA. Hence, if the system selects correctly, we can assume that the effect of the test scores over the GPA in non-enrolled applicants should be lower than the one in the enrolled applicants. Thus, we suppose that the impact of X over Y in the observed group, $\{Z = i, S \neq 0\}$, is at least equal to the one in the non-observed group, $\{Z = i, S = 0\}$. This assumption translates to: the marginal effect in each non-observed group, i, can not be higher than the maximum observed marginal effect into the same $Z = i$. This identification restriction allow us to make more informative the upper bound for the non-observed derivatives, such that:

$$0 \leq \left.\frac{d\mathbb{E}(Y|X, Z = i, S = 0)}{dX}\right|_{X=x} \leq \max_{x \in \mathcal{X}_{ji}} \left\{ \left.\frac{d\mathbb{E}(Y|X, Z = i, S \neq 0)}{dX}\right|_{X=x} \right\}.$$

Under this assumption we have that $D_{0ix} = 0$ and D_{1ix} is a function that depends on the observed marginal effects in each $Z = i$, as follows:

$$D_{1ix} := \begin{cases} \max\limits_{x\in\mathcal{X}_{ii}} \left\{ \frac{d\mathbb{E}(Y|X,Z=i,S=i)}{dX}\Big|_{X=x} \right\} & \text{if } i \in \{1,2,3\} \\ \min\limits_{x\in\{\mathcal{X}_{14}\cup\mathcal{X}_{24}\}} \left\{ \max\limits_{x\in\mathcal{X}_{14}} \left\{ \frac{dE_{14}(X)}{dX}\Big|_{X=x} \right\}, \max\limits_{x\in\mathcal{X}_{24}} \left\{ \frac{dE_{24}(X)}{dX}\Big|_{X=x} \right\} \right\} & \text{if } i = 4 \\ \min\limits_{x\in\{\mathcal{X}_{15}\cup\mathcal{X}_{35}\}} \left\{ \max\limits_{x\in\mathcal{X}_{15}} \left\{ \frac{dE_{15}(X)}{dX}\Big|_{X=x} \right\}, \max\limits_{x\in\mathcal{X}_{35}} \left\{ \frac{dE_{35}(X)}{dX}\Big|_{X=x} \right\} \right\} & \text{if } i = 5 \\ \min\limits_{x\in\{\mathcal{X}_{26}\cup\mathcal{X}_{36}\}} \left\{ \max\limits_{x\in\mathcal{X}_{26}} \left\{ \frac{dE_{26}(X)}{dX}\Big|_{X=x} \right\}, \max\limits_{x\in\mathcal{X}_{36}} \left\{ \frac{dE_{36}(X)}{dX}\Big|_{X=x} \right\} \right\} & \text{if } i = 6 \\ \min\limits_{x\in\{\mathcal{X}_{17}\cup\mathcal{X}_{27}\cup\mathcal{X}_{37}\}} \left\{ \max\limits_{x\in\mathcal{X}_{17}} \left\{ \frac{dE_{17}(X)}{dX}\Big|_{X=x} \right\}, \max\limits_{x\in\mathcal{X}_{27}} \left\{ \frac{dE_{27}(X)}{dX}\Big|_{X=x} \right\}, \right. & \\ \left. \max\limits_{x\in\mathcal{X}_{37}} \left\{ \frac{dE_{37}(X)}{dX}\Big|_{X=x} \right\} \right\} & \text{if } i = 7 \end{cases}$$

Hence,

$$\mathbb{E}(D_{0x}|X=x,S=0) = 0,$$

and

$$\begin{aligned} \mathbb{E}(D_{1x}|X=x,S=0) &= \sum_{i=1}^{3} \max_{x\in\mathcal{X}_{ii}} \left\{ \frac{d\mathbb{E}(Y|X,Z=i,S=i)}{dX}\Big|_{X=x} \right\} \mathbb{P}(Z=i|X=x,S=0) \\ &+ \min_{x\in\{\mathcal{X}_{14}\cup\mathcal{X}_{24}\}} \left\{ \max_{x\in\mathcal{X}_{14}} \left\{ \frac{dE_{14}(X)}{dX}\Big|_{X=x} \right\}, \max_{x\in\mathcal{X}_{24}} \left\{ \frac{dE_{24}(X)}{dX}\Big|_{X=x} \right\} \right\} \\ &\quad \times \mathbb{P}(Z=4|X=x,S=0) \\ &+ \min_{x\in\{\mathcal{X}_{15}\cup\mathcal{X}_{35}\}} \left\{ \max_{x\in\mathcal{X}_{15}} \left\{ \frac{dE_{15}(X)}{dX}\Big|_{X=x} \right\}, \max_{x\in\mathcal{X}_{35}} \left\{ \frac{dE_{35}(X)}{dX}\Big|_{X=x} \right\} \right\} \\ &\quad \times \mathbb{P}(Z=5|X=x,S=0) \\ &+ \min_{x\in\{\mathcal{X}_{26}\cup\mathcal{X}_{36}\}} \left\{ \max_{x\in\mathcal{X}_{26}} \left\{ \frac{dE_{26}(X)}{dX}\Big|_{X=x} \right\}, \max_{x\in\mathcal{X}_{36}} \left\{ \frac{dE_{36}(X)}{dX}\Big|_{X=x} \right\} \right\} \\ &\quad \times \mathbb{P}(Z=6|X=x,S=0) \\ &+ \min_{x\in\{\mathcal{X}_{17}\cup\mathcal{X}_{27}\cup\mathcal{X}_{37}\}} \left\{ \max_{x\in\mathcal{X}_{17}} \left\{ \frac{dE_{17}(X)}{dX}\Big|_{X=x} \right\}, \max_{x\in\mathcal{X}_{27}} \left\{ \frac{dE_{27}(X)}{dX}\Big|_{X=x} \right\}, \right. \\ &\qquad \left. \max_{x\in\mathcal{X}_{37}} \left\{ \frac{dE_{37}(X)}{dX}\Big|_{X=x} \right\} \right\} \mathbb{P}(Z=7|X=x,S=0) \end{aligned}$$

Regarding both y_{0ix} and y_{1ix} they must be chosen according to the Scenario 1 in Section 3.2.

Scenario 2: The system selects wrongly

In contrast to the Scenario 1, to assume that the selection process is wrong, is equivalent to assume that if only information about the scores is used, then the predictability of the GPA in the non-observed group will be better than the one in the observed group. Thus, we suppose that the impact of X over Y in the observed group, $\{Z = i, S \neq 0\}$, is at most equal to the one in the non-observed group, $\{Z = i, S = 0\}$. This assumption translates to: the marginal effect in each non-observed group, i, is higher than the maximum observed marginal effect into the same $Z = i$. This identification restriction allow us to make more informative the lower bound for the non-observed derivatives, such that:

$$\max_{x\in\mathcal{X}_{ji}}\left\{\left.\frac{d\mathbb{E}(Y|X,Z=i,S\neq 0)}{dX}\right|_{X=x}\right\} \leq \left.\frac{d\mathbb{E}(Y|X,Z=i,S=0)}{dX}\right|_{X=x} \leq \max_{x\in\mathcal{X}}\left\{\left.\frac{d\mathbb{E}(Y|X,S\neq 0)}{dX}\right|_{X=x}\right\}.$$

Under this assumption we have that D_{0ix} is a function that depends on the observed marginal effects in each $Z = i$, as follows:

$$D_{0ix} := \begin{cases} \max\limits_{x\in\mathcal{X}_{ii}}\left\{\left.\frac{d\mathbb{E}(Y|X,Z=i,S=i)}{dX}\right|_{X=x}\right\} & \text{if } i\in\{1,2,3\} \\ \max\limits_{x\in\{\mathcal{X}_{14}\cup\mathcal{X}_{24}\}}\left\{\max\limits_{x\in\mathcal{X}_{14}}\left\{\left.\frac{dE_{14}(X)}{dX}\right|_{X=x}\right\}, \max\limits_{x\in\mathcal{X}_{24}}\left\{\left.\frac{dE_{24}(X)}{dX}\right|_{X=x}\right\}\right\} & \text{if } i=4 \\ \max\limits_{x\in\{\mathcal{X}_{15}\cup\mathcal{X}_{35}\}}\left\{\max\limits_{x\in\mathcal{X}_{15}}\left\{\left.\frac{dE_{15}(X)}{dX}\right|_{X=x}\right\}, \max\limits_{x\in\mathcal{X}_{35}}\left\{\left.\frac{dE_{35}(X)}{dX}\right|_{X=x}\right\}\right\} & \text{if } i=5 \\ \max\limits_{x\in\{\mathcal{X}_{26}\cup\mathcal{X}_{36}\}}\left\{\max\limits_{x\in\mathcal{X}_{26}}\left\{\left.\frac{dE_{26}(X)}{dX}\right|_{X=x}\right\}, \max\limits_{x\in\mathcal{X}_{36}}\left\{\left.\frac{dE_{36}(X)}{dX}\right|_{X=x}\right\}\right\} & \text{if } i=6 \\ \max\limits_{x\in\{\mathcal{X}_{17}\cup\mathcal{X}_{27}\cup\mathcal{X}_{37}\}}\left\{\max\limits_{x\in\mathcal{X}_{17}}\left\{\left.\frac{dE_{17}(X)}{dX}\right|_{X=x}\right\}, \max\limits_{x\in\mathcal{X}_{27}}\left\{\left.\frac{dE_{27}(X)}{dX}\right|_{X=x}\right\}, \right. & \\ \left. \max\limits_{x\in\mathcal{X}_{37}}\left\{\left.\frac{dE_{37}(X)}{dX}\right|_{X=x}\right\}\right\} & \text{if } i=7 \end{cases}$$

Regarding D_{1ix}, we use the proposed identification restriction as well as in (3.8). Hence, we are assuming that if the system selects wrongly, the maximum effect that X has over Y is the maximum observed one. Hence,

$$
\begin{aligned}
\mathbb{E}(D_{0x}|X=x,S=0) &= \sum_{i=1}^{3} \max_{x\in\mathcal{X}_{ii}} \left\{ \frac{d\mathbb{E}(Y|X,Z=i,S=i)}{dX}\bigg|_{X=x} \right\} \mathbb{P}(Z=i|X=x,S=0) \\
&+ \max_{x\in\{\mathcal{X}_{14}\cup\mathcal{X}_{24}\}} \left\{ \max_{x\in\mathcal{X}_{14}} \left\{ \frac{dE_{14}(X)}{dX}\bigg|_{X=x} \right\}, \max_{x\in\mathcal{X}_{24}} \left\{ \frac{dE_{24}(X)}{dX}\bigg|_{X=x} \right\} \right\} \\
&\quad \times \mathbb{P}(Z=4|X=x,S=0) \\
&+ \max_{x\in\{\mathcal{X}_{15}\cup\mathcal{X}_{35}\}} \left\{ \max_{x\in\mathcal{X}_{15}} \left\{ \frac{dE_{15}(X)}{dX}\bigg|_{X=x} \right\}, \max_{x\in\mathcal{X}_{35}} \left\{ \frac{dE_{35}(X)}{dX}\bigg|_{X=x} \right\} \right\} \\
&\quad \times \mathbb{P}(Z=5|X=x,S=0) \\
&+ \max_{x\in\{\mathcal{X}_{26}\cup\mathcal{X}_{36}\}} \left\{ \max_{x\in\mathcal{X}_{26}} \left\{ \frac{dE_{26}(X)}{dX}\bigg|_{X=x} \right\}, \max_{x\in\mathcal{X}_{36}} \left\{ \frac{dE_{36}(X)}{dX}\bigg|_{X=x} \right\} \right\} \\
&\quad \times \mathbb{P}(Z=6|X=x,S=0) \\
&+ \max_{x\in\{\mathcal{X}_{17}\cup\mathcal{X}_{27}\cup\mathcal{X}_{37}\}} \left\{ \max_{x\in\mathcal{X}_{17}} \left\{ \frac{dE_{17}(X)}{dX}\bigg|_{X=x} \right\}, \max_{x\in\mathcal{X}_{27}} \left\{ \frac{dE_{27}(X)}{dX}\bigg|_{X=x} \right\}, \right. \\
&\qquad \left. \max_{x\in\mathcal{X}_{37}} \left\{ \frac{dE_{37}(X)}{dX}\bigg|_{X=x} \right\} \right\} \mathbb{P}(Z=7|X=x,S=0),
\end{aligned}
$$

and

$$
\mathbb{E}(D_{1x}|X=x,S=0) = \max_{x\in\mathcal{X}} \left\{ \frac{d\mathbb{E}(Y|X,S\neq 0)}{dX}\bigg|_{X=x} \right\}.
$$

Regarding both y_{0ix} and y_{1ix} they must be chosen according to the Scenario 2 in Section 3.2.

Additionally to the beliefs about the correctness in the selection process, we used the the purpose of the test: to select the "best applicants" in some specific sense (Alarcón-Bustamante et al., 2020). This fact allow us to assume that the conditional expectation in the non-observed group is a non-decreasing function of the scores and, consequently its derivative will will be greater or equal to zero.

Analogously to the identification bounds for the conditional expectation, we used the two mentioned scenarios to find the ones for the marginal effect. Note that assumptions about both the not-identified conditional expectation and its derivative are combined to find the identification bounds. Hence, regarding the proposed identification bounds in Remarks 3.3.1 and 3.3.2, this combination will help to make more informative both the lower and the upper bounds for the marginal effect. However, the Scenario 1 allow us to make even more informative the upper bound, while the Scenario 2 makes even more informative the lower one.

3.4 Results from the case-study

In this section we show the results about the identification bounds for both the expected GPA and the marginal effect in each considered selection factor. Firstly, we explain how the bounds were estimated. Secondly, we show the plots for the identification bounds for each considered Scenario.

3.4.1 Estimation of the bounds

The estimation of the bounds for the conditional expectation is in two steps. Firstly, based on the Nadaraya-Watson Estimator (Nadaraya, 1964; Watson, 1964), and the Kernel density estimator a Gaussian Kernel was used for all the groups in $\mathbb{E}(Y|X, Z = i, S \neq 0), \mathbb{P}(S = j|X, Z = i)$, and $\mathbb{P}(Z = i|X)$. Regarding the window width, it was computed by using the `density` function from the `stats` R-package (Wang, 2010). Secondly, the estimation in step one is plugged in the bounds. In order to obtain the wider bound, from Remark 3.2.1, and by taking into account that in Chile a score of 1 is the minimum GPA that could be obtained, and a score of 7 the maximum one, we evaluated our method using $y_0 = 1$ and $y_1 = 7$. For the identification bounds of the marginal effects, we computed the involved derivatives by using finite differences.

3.4.2 Results

We show the identification bounds for both the conditional expectation and the global marginal effect (see Alarcón-Bustamante et al., 2021) for each selection factor in both scenarios *the System selects correctly* and *the System selects wrongly*. The grey bounds represent the theoretical ones as well as in Remarks (3.2.1) and (3.3.1). The blue bounds represent the empirical ones as well as in Remarks (3.2.2) and (3.3.2). The green bounds represent the proposed identification bounds. The dashed lines in the identification bounds for the conditional expectation represent the upper bound, while the the solid one represent the lower bound. Regarding the bullet points in the identification bounds for the marginal effect, they represents the points where the function has derivative. Finally, the red-solid line represents the estimation of both the conditional expectation and the marginal effect of X by using a Multiple Regression Model, the traditional procedure that have been used in Chile in order to evaluate the predictive capacity of the selection factors.

As it was argued, the severity of the identification problem is with respect to lower scores. In

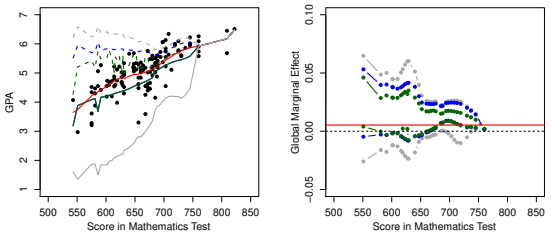
fact, for lower scores in a selection factor, we find high probabilities of not being enrolled in a program. When all the bounds coincide we say that we do not have non-enrolled students with these scores (see identification bounds for both the conditional expectation and the marginal effect of Mathematics, Language, Sciences, and HGPA score). However, for the Ranking score we can see that the identification problem is for higher scores also, and therefore non-enrolled students with the maximum Ranking score are not enrolled.

The System selects correctly

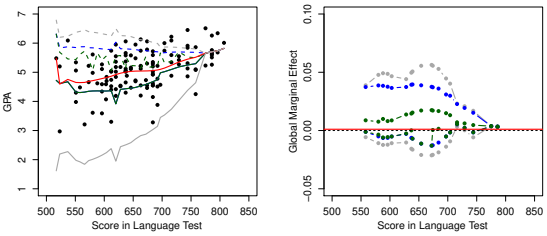
The identification bounds give us a range of all the plausible values of the evaluated function if a researcher is ready to believe in the proposed identification restrictions. In this context, if we are ready to believe that the score in the selection factor is the unique one that can predict the performance in the GPA, to predict the GPA with the available information only is a plausible solution for learning about the expected GPA, i.e., the regression $\mathbb{E}(Y|X, S \neq 0)$ under ignorability is in between of the bounds in all the cases.

Regarding the marginal effect, we can see that if the Scenario 1 is considered (i.e., we assume that selection factors are the unique predictor for the performance in the university, and the selection factor is such that high scores will produce high performance) if non-enrolled students had enrolled, a plausible scenario is that they were caused a null effect of the scores over the GPA. However, if students with scores greater than 700 in mathematics test would be enrolled, the marginal effect would be positive (see the identification bounds for the Global Marginal Effect in Figures 3.3a, 3.3d, and 3.3c).

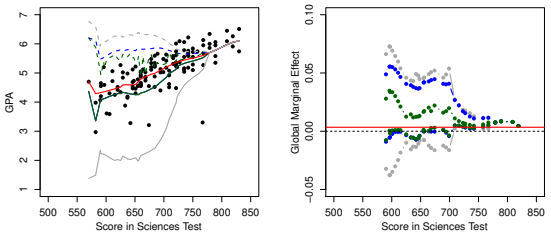
By considering the interpretation of the marginal effect in partitioned populations provided in Alarcón-Bustamante et al. (2021), we can see that if in the non-enrolled students the predictability of the scores over the GPA is equal to the maximum one in each group, then the global marginal effect of score in Mathematics, Sciences and HGPA score has a tendency of being a decreasing function of the scores. Thus, if Scenario 1 holds, for lower score values, there is a larger proportion of students belonging to a program where the marginal effect of Mathematics and Sciences test score, and HGPA score is high. In contrast, for higher scores, a larger proportion of students will be found for a program where the marginal effect is low. Regarding to the global marginal effect of score in Language, and Ranking score, they have a tendency of being a constant function of the



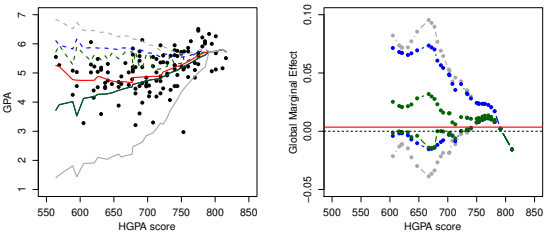
(a) Identification bounds in the Mathematics test



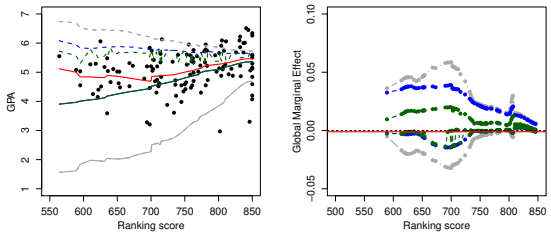
(b) Identification bounds in the Language and Communication test



(c) Identification bounds in the Sciences test



(d) Identification bounds in HGPA selection factor



(e) Identification bounds in Ranking selection factor

Figure (3.3) Identification bounds for both the conditional expectation (left-side) and the Marginal Effect (right-side) assuming that the System selects correctly.

scores. Thus, if Scenario 1 holds, for lower and upper score values, there is the same proportion of students belonging to a program where the marginal effect of Language test score, and Ranking score is high or low.

Note that the lower bound is near to be equal to the marginal effect when the traditional procedure is used in Chile to evaluate the predictive capacity of the selection factors. Hence, if it assumed

both that the scores of selection factors in non-enrolled students is zero and that Scenario 1 holds, the enrolment of non-enrolled applicants would have produced a null global marginal effect for all the selection factors.

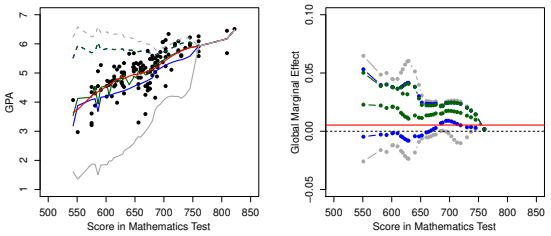
The System selects wrongly

As it was mentioned before, assuming that the system selects wrongly, is analogous to assume that the selection factors are not the unique ones that can predict the performance in the GPA. In this context, we can think that if an applicant that was not-selected by the system had been selected, its performance would be better than a selected one. This identification restriction allowed making more informative the lower bound for both the conditional expectation and the marginal effect.

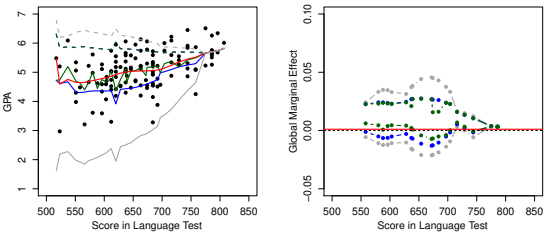
Regarding the first one, for all the selection factors the lower bound is practically equivalent to the regression line under the ignorability assumption. Thus, if we are ready to believe that the expected GPA, given the test scores, in non-enrolled applicants had been at least equal to the ones in enrolled students we obtain that $\mathbb{E}(Y|X) \approx \mathbb{E}(Y|X, S \neq 0)$ (see the lower bound for the conditional expectation in Figures 3.4a, 3.4b, and 3.4c). For the HGPA score, this fact is reflected in scores higher than 600, and for the Ranking score, higher that 700. In this context, the ignorability assumption could reflect that the system selects wrongly because a plausible solution of the regression when this scenario is assumed (the lower bound) is equal to the regression under ignorability.

Regarding the Global Marginal Effect, the lower bound in Mathematics and Science tests are higher than the one when the using Multiple Regression Model is used for learning about the effect of the selection factor scores over the GPA. Hence, if we assume that the impact of scores over the GPA in the observed group, $Z = i, S \neq 0$, is at most equal to the one in the non-observed group, $Z = i, S = 0$, the marginal effect using Multiple Regression Model is not a plausible solution.

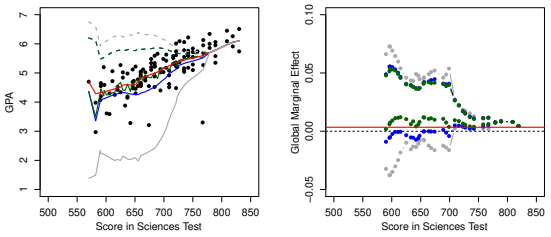
Taking into account the interpretation of the Global Marginal Effect, we have that its tendency is near to be constant, thus for lower or higher scores, we will find in the same proportion students in both belonging to a program where the marginal effect is high and belonging to a program where the marginal effect is low.



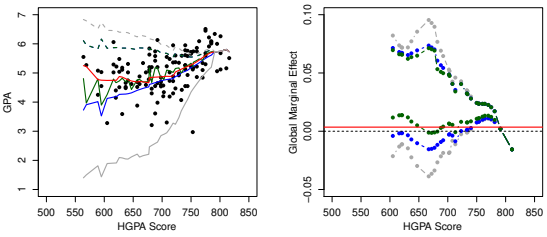
(a) Identification bounds in Mathematics test



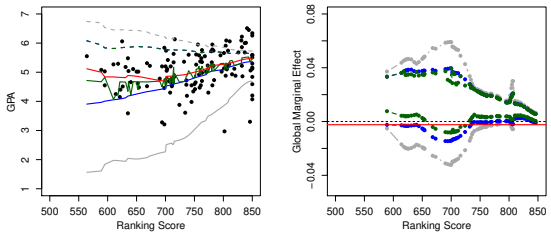
(b) Identification bounds in Language and Communication test



(c) Identification bounds in Sciences test



(d) Identification bounds in HGPA selection factor



(e) Identification bounds in Ranking selection factor

Figure (3.4) Identification bounds for both the conditional expectation (left-side) and the Marginal Effect (right-side) assuming that the System selects wrongly.

3.5 Conclusions and discussion

We have used a partial identification approach for learning about both the conditional expectation and the marginal effect of test scores over a variable of interest. From a modelling perspective, we have generalised the Global Marginal Effect definition of Alarcón-Bustamante et al. (2021) to

the case of partial observability of the outcome. This was carried out by making a partition of the whole population of the applicants, and not assuming a prior distributional knowledge about the missing GPAs. In fact, we considered not only the desired properties of a selection test but also the possible Scenarios in the selection process in Chile. More specifically, we used monotonicity assumptions in order to find the set of all the possible values of the marginal effect by considering that the selection process is correct (or wrong). In this context, the main contribution of this chapter is to show how a few ideas can help us to make inferences over the conditional expectation and its derivative in presence of missing values.

Although the regression line, under the ignorability assumption is a plausible solution when Scenario 1 is assumed, results illustrate that the current solution for learning about it is closer to reflect that the system selects wrongly than the one selects correctly. Regarding the Marginal effect,

The low predictive capacity of the selection factors have caused multiple changes in the battery tests in Chile (see DEMRE, 2016; Grassau, 1956; Donoso, 1998). Nevertheless, our results show that assuming the only the proposed selection factors in Chile are useful to select is only a plausible solution for learning about the predictive validity of the selection tests. Moreover, if other selection factors were used, the predictive capacity of the tests would be maybe better.

Chapter 4

Final conclusions and remarks

"The credibility of inferences decreases with the strength of the assumptions maintained."
Charles Manski: The Law of Decreasing Credibility.

Jerry Hausman, Professor of Economics at MIT in Cambridge, Massachusetts, USA, stated to Charles Manski: *you can not give the client a bound. The client needs a point*. Nevertheless, point predictions are produced by imposing strong assumptions that are rarely funded (Manski, 2013).

This work intends to propose how to draw conclusions based on both the problem and the population of interest. Although the used assumptions are not testable, they are weaker, and more consistent, than the ones described in the introduction of this dissertation. Our interest was to define a new way to analyse the predictive validity of a selection test by considering all the available information (about the applicants and the selected ones). Our partial identification methodology is based on both a desired property of a selection test and the credibility about how the system selects. This fact allows to interpret both the conditional expectation and the marginal effect according to the credibility that a policy maker has about the chosen hypothesis.

Additionally to propose a new way to analyse and interpret a marginal effect for the case of partitioned populations, we extended the idea of identification bounds for the effect of changes in the explanartory random variable proposed by Manski (1989) to the case of multiple sampling processes.

Even though the approach considered in this dissertation extended and tackled some drawbacks of traditional strategies, there are some open questions to be discussed, which are listed bellow:

1. This work make sense when exist a partial observability of the outcome only. Thus, we assume that the covariates are fully observed. Nevertheless, in Chile there are elective tests, and therefore some applicants take one and not the another. Thus, a relevant future work is to consider different missing-data patterns and combine the results of Manski (2007) and the ones of this dissertation.

2. A natural extension of this work is to consider the case where the conditional expectation depends on more than one covariate.

3. In order to have a whole overview of the selection system in Chile, to generalise the partition in Chapter 3 to more than three groups is considered as future work.

To finish this dissertation, we highlight that there is a not unique way to model a situation (e.g., by using the traditional distributional assumptions). In fact, we used assumptions related to the problem that we intend to model. Finally, from our viewpoint, there is a necessity to make people known how to deal with missing values and how *stronger assumptions yield conclusions that are more powerful but less credible* (The Law of Decreasing Credibility Manski, 2003, p. 1).

Appendices

A Proof of the invariant property of the Global Marginal Effect

Proof. The global marginal effect in equation (2.2) is a function that depends on both X and Z, namely $g(X, Z)$. Let z' be any chosen reference group. Noting that $p_{z'}(X) = 1 - \sum_{z \neq z'} p_z(X)$, where $p_z(X) = \mathbb{P}(Z = z|X)$ we have

$$\begin{aligned} g(X, Z = z') &= \frac{d\mathbb{E}(Y|X, Z = z')}{dX} + \sum_{z \neq z'} \left[\frac{d\mathbb{E}(Y|X, Z = z)}{dX} - \frac{d\mathbb{E}(Y|X, Z = z')}{dX} \right] p_z(X) \\ &+ \sum_{z \neq z'} \left[E(Y|X, Z = z) - \mathbb{E}(Y|X, Z = z') \right] \frac{dp_z(X)}{dX} \end{aligned}$$

Now, suppose that another reference group, z'' ($z' \neq z''$), is chosen. Then,

$$\begin{aligned} g(X, Z = z'') &= \frac{d\mathbb{E}(Y|X, Z = z'')}{dX} + \sum_{z \neq z''} \left[\frac{d\mathbb{E}(Y|X, Z = z)}{dX} - \frac{d\mathbb{E}(Y|X, Z = z'')}{dX} \right] p_z(X) \\ &+ \sum_{z \neq z''} \left[E(Y|X, Z = z) - \mathbb{E}(Y|X, Z = z'') \right] \frac{dp_z(X)}{dX} \end{aligned}$$

By subtracting both functions we obtain

$$\begin{aligned} g(X, Z = z') - g(X, Z = z'') &= \left[\frac{d\mathbb{E}(Y|X, Z = z'')}{dX} - \frac{d\mathbb{E}(Y|X, Z = z')}{dX} \right] \left[\sum_z p_z(X) - 1 \right] \\ &+ \left[E(Y|X, Z = z'') - \mathbb{E}(Y|X, Z = z') \right] \frac{d\left[\sum_z p_z(X) \right]}{dX} \end{aligned}$$

Finally, because $\sum_z p_z(X) = 1$, hence

$$\left[\sum_z p_z(X) - 1 \right] = 0 \quad ; \quad \frac{d\left[\sum_z p_z(X) \right]}{dX} = 0$$

This fact implies that

$$g(X, Z = z') - g(X, Z = z'') = 0,$$

and therefore $g(X, Z = z') = g(X, Z = z'')$, for all $z' \neq z''$. □

Remark A.1. *Analogously, it can be proven that $g(X, Z)$ is invariant under the chosen reference group when $Z \perp\!\!\!\perp X$.*

B Proof Proposition 3.2.1

Proof. Let us suppose that $y_{0ix} \leq \mathbb{E}(Y|X = x, Z = i, S = 0) \leq y_{1ix}$, with $y_{0ix} \leq y_{1ix}$. Multiplying this inequality by $\mathbb{P}(S = 0, Z = i|X = x)$, we obtain that

$$y_{0ix}\mathbb{P}(S = 0, Z = i|X = x) \leq \mathbb{E}(Y|X = x, Z = i, S = 0)\mathbb{P}(S = 0, Z = i|X = x) \leq$$
$$y_{1ix}\mathbb{P}(S = 0, Z = i|X = x).$$

Adding the term $\sum_{j=1}^{3} \mathbb{E}(Y|X = x, Z = i, S = j)\mathbb{P}(S = j, Z = i|X = x)$ to this inequality, we obtain that:

$$\sum_{i=1}^{7} y_{0ix}\mathbb{P}(Z = i|X = x, S = 0)\mathbb{P}(S = 0|X = x) + \sum_{i=1}^{7}\sum_{j=1}^{3} E_{ji}(x)p_{ji}(x) \leq \mathbb{E}(Y|X = x)$$
$$\leq \sum_{i=1}^{7} y_{1ix}\mathbb{P}(Z = i|X = x, S = 0)\mathbb{P}(S = 0|X = x) + \sum_{i=1}^{7}\sum_{j=1}^{3} E_{ji}(x)p_{ji}(x).$$

Provided that $\sum_{i=1}^{7} \mathbb{P}(Z = i|X = x, S = 0) = 1$, and that we defined $Y_{0x} = (y_{01x}, \cdots, y_{07x})$, and $Y_{1x} = (y_{11x}, \cdots, y_{17x})$ we have that

- $\mathbb{E}(Y_{0x}|X = x, S = 0) = \sum_{i=1}^{7} y_{0ix}\mathbb{P}(Z = i|X = x, S = 0)$, and
- $\mathbb{E}(Y_{1x}|X = x, S = 0) = \sum_{i=1}^{7} y_{1ix}\mathbb{P}(Z = i|X = x, S = 0)$

□

C Proof Proposition 3.3.1

Let $\frac{d^+\lambda(X)}{dX}\Big|_{X=x}$ and $\frac{d^-\lambda(X)}{dX}\Big|_{X=x}$ be the positive and negative part of $\frac{d\lambda(X)}{dX}\Big|_{X=x}$ respectively, such that:

$$\frac{d^+\lambda(X)}{dX}\Big|_{X=x} = \max\left\{0, \frac{d\lambda(X)}{dX}\Big|_{X=x}\right\} = \begin{cases} \frac{d\lambda(X)}{dX}\Big|_{X=x} & \text{if } \frac{d\lambda(X)}{dX}\Big|_{X=x} > 0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\frac{d^-\lambda(X)}{dX}\Big|_{X=x} = \max\left\{0, -\frac{d\lambda(X)}{dX}\Big|_{X=x}\right\} = \begin{cases} -\frac{d\lambda(X)}{dX}\Big|_{X=x} & \text{if } \frac{d\lambda(X)}{dX}\Big|_{X=x} < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Where $\frac{d\lambda(X)}{dX}\Big|_{X=x} = \frac{d^+\lambda(X)}{dX}\Big|_{X=x} - \frac{d^-\lambda(X)}{dX}\Big|_{X=x}$, and $\left|\frac{d\lambda(X)}{dX}\Big|_{X=x}\right| = \frac{d^+\lambda(X)}{dX}\Big|_{X=x} + \frac{d^-\lambda(X)}{dX}\Big|_{X=x}$.

Proof. Equation (3.5) can be written as:

$$\begin{aligned} ME^X = & \sum_{i=1}^{7}\left[\frac{d\mathbb{E}(Y|X,Z=i,S=0)}{dX}\mathbb{P}(S=0|X,Z=i)\right]\mathbb{P}(Z=i|X) \\ & + \sum_{i=1}^{7}\left[\sum_{j=1}^{3}\left[\mathbb{E}(Y|X,Z=i,S=j) - \mathbb{E}(Y|X,Z=i,S=0)\right]\frac{d\mathbb{P}(S=j|X,Z=i)}{dX}\right]\mathbb{P}(Z=i|X) \\ & + \sum_{i=1}^{7}\mathbb{E}(Y|X,Z=i,S=0)\mathbb{P}(S=0|X,Z=i)\frac{d\mathbb{P}(Z=i|X)}{dX} \\ & + \sum_{i=1}^{7}\left[\sum_{j=1}^{3}\frac{d\mathbb{E}(Y|X,Z=i,S=j)}{dX}\mathbb{P}(S=j|X,Z=i)\right]\mathbb{P}(Z=i|X) \\ & + \sum_{i=1}^{7}\left[\sum_{j=1}^{3}\mathbb{E}(Y|X,Z=i,S=j)\mathbb{P}(S=j|X,Z=i)\right]\frac{d\mathbb{P}(Z=i|X)}{dX}. \end{aligned} \tag{1}$$

Because of both $\mathbb{E}(Y|X,Z=i,S=0)$ and its derivative are not identified by data generating process, the first three sums in (1) are not identified.

Let us suppose that $D_{0ix} \leq \frac{d\mathbb{E}(Y|X,Z=i,S=0)}{dX} \leq D_{1ix}$, thus for each $X=x$ the first sum is

bounded as follows:

$$\sum_{i=1}^{7} D_{0ix}\mathbb{P}(S=0|X=x,Z=i)\mathbb{P}(Z=i|X=x) \leq$$

$$\sum_{i=1}^{7}\left[\left.\frac{d\mathbb{E}(Y|X,Z=i,S=0)}{dX}\right|_{X=x}\mathbb{P}(S=0|X=x,Z=i)\right]\mathbb{P}(Z=i|X=x) \leq$$

$$\sum_{i=1}^{7} D_{1ix}\mathbb{P}(S=0|X=x,Z=i)\mathbb{P}(Z=i|X=x),$$

which it can be written as

$$\left[\sum_{i=1}^{7} D_{0ix}\,\mathbb{P}(Z=i|X=x,S=0)\right]\mathbb{P}(S=0|X=x) \leq$$

$$\sum_{i=1}^{7}\left[\left.\frac{d\mathbb{E}(Y|X,Z=i,S=0)}{dX}\right|_{X=x}\mathbb{P}(S=0|X=x,Z=i)\right]\mathbb{P}(Z=i|X=x) \leq$$

$$\left[\sum_{i=1}^{7} D_{1ix}\mathbb{P}(Z=i|X=x,S=0)\right]\mathbb{P}(S=0|X=x),$$

Provided that $\sum_{i=1}^{7}\mathbb{P}(Z=i|X=x,S=0)=1$, we have that:

$$\mathbb{E}(D_{0x}|X=x,S=0)\mathbb{P}(S=0|X=x) \leq$$

$$\sum_{i=1}^{7}\left[\left.\frac{d\mathbb{E}(Y|X,Z=i,S=0)}{dX}\right|_{X=x}\mathbb{P}(S=0|X=x,Z=i)\right]\mathbb{P}(Z=i|X=x) \leq$$

$$\mathbb{E}(D_{1x}|X=x,S=0)\mathbb{P}(S=0|X=x), \tag{2}$$

Suppose that the non-observed conditional expectation is bounded by $y_{0ix} \leq \mathbb{E}(Y|X,Z=i,S=0) \leq y_{1ix}$. Thus, for each $X=x$, the identification region for the positive part of

$$\sum_{j=1}^{3}\left[\mathbb{E}(Y|X=x,Z=i,S=j)-\mathbb{E}(Y|X=x,Z=i,S=0)\right]\left.\frac{d\mathbb{P}(S=j|X,Z=i)}{dX}\right|_{X=x}$$

is bounded by

$$\sum_{j=1}^{3}\Big[\mathbb{E}(Y|\ X=x,Z=i,S=j)-y_{1ix}]\ \frac{d^{+}\mathbb{P}(S=j|X,Z=i)}{dX}\Big|_{X=x}\leq$$
$$\sum_{j=1}^{3}[\mathbb{E}(Y|X=x,Z=i,S=j)-\mathbb{E}(Y|X=x,Z=i,S=0)]\ \frac{d^{+}\mathbb{P}(S=j|X,Z=i)}{dX}\Big|_{X=x}\leq$$
$$\sum_{j=1}^{3}[\mathbb{E}(Y|X=x,Z=i,S=j)-y_{0ix}]\ \frac{d^{+}\mathbb{P}(S=j|X,Z=i)}{dX}\Big|_{X=x}. \tag{3}$$

Analogously, the negative one is bounded by:

$$\sum_{j=1}^{3}\Big[\mathbb{E}(Y|\ X=x,Z=i,S=j)-y_{1ix}]\ \frac{d^{-}\mathbb{P}(S=j|X,Z=i)}{dX}\Big|_{X=x}\leq$$
$$\sum_{j=1}^{3}[\mathbb{E}(Y|X=x,Z=i,S=j)-\mathbb{E}(Y|X=x,Z=i,S=0)]\ \frac{d^{-}\mathbb{P}(S=j|X,Z=i)}{dX}\Big|_{X=x}\leq$$
$$\sum_{j=1}^{3}[\mathbb{E}(Y|X=x,Z=i,S=j)-y_{0ix}]\ \frac{d^{-}\mathbb{P}(S=j|X,Z=i)}{dX}\Big|_{X=x}. \tag{4}$$

By subtracting (4) from (3), we obtain that

$$f_{1i}^{+}(x)-f_{0i}^{-}(x)\leq$$
$$\sum_{j=1}^{3}[\mathbb{E}(Y|X=x,Z=i,S=j)-\mathbb{E}(Y|X=x,Z=i,S=0)]\ \frac{d\mathbb{P}(S=j|X,Z=i)}{dX}\Big|_{X=x}\leq$$
$$f_{0i}^{+}(x)-f_{1i}^{-}(x)$$

Provided that $\sum_{i=1}^{7}\mathbb{P}(Z=i|X=x)=1$, we have that:

$$\mathbb{E}(f_{1}^{+}(X)-f_{0}^{-}(X)|X=x)\leq$$
$$\sum_{i=i}^{7}\left\{\sum_{j=1}^{3}[\mathbb{E}(Y|X=x,Z=i,S=j)-\mathbb{E}(Y|X=x,Z=i,S=0)]\ \frac{d\mathbb{P}(S=j|X,Z=i)}{dX}\Big|_{X=x}\right\}$$
$$\times\mathbb{P}(Z=i|X=x)\leq\mathbb{E}(f_{0}^{+}(X)-f_{1}^{-}(X)|X=x). \tag{5}$$

Analogously to the previous identification bound, for each $X = x$, the positive part of

$$\sum_{i=1}^{7} \mathbb{E}(Y|X, Z = i, S = 0)\mathbb{P}(S = 0|X, Z = i)\frac{d\mathbb{P}(Z = i|X)}{dX}$$

is bounded as follows:

$$\sum_{i=1}^{7} y_{0ix} \left.\frac{d^{+}\mathbb{P}(Z = i|X)}{dX}\right|_{X=x} \mathbb{P}(S = 0|X = x, Z = i) \leq$$

$$\sum_{i=1}^{7} \mathbb{E}(Y|X = x, Z = i, S = 0)\mathbb{P}(S = 0|X = x, Z = i) \left.\frac{d^{+}\mathbb{P}(Z = i|X)}{dX}\right|_{X=x} \leq$$

$$\sum_{i=1}^{7} y_{1ix} \left.\frac{d^{+}\mathbb{P}(Z = i|X)}{dX}\right|_{X=x} \mathbb{P}(S = 0|X = x, Z = i), \tag{6}$$

and the negative one is bounded by

$$\sum_{i=1}^{7} y_{0ix} \left.\frac{d^{-}\mathbb{P}(Z = i|X)}{dX}\right|_{X=x} \mathbb{P}(S = 0|X = x, Z = i) \leq$$

$$\sum_{i=1}^{7} \mathbb{E}(Y|X = x, Z = i, S = 0)\mathbb{P}(S = 0|X = x, Z = i) \left.\frac{d^{-}\mathbb{P}(Z = i|X)}{dX}\right|_{X=x} \leq$$

$$\sum_{i=1}^{7} y_{1ix} \left.\frac{d^{-}\mathbb{P}(Z = i|X)}{dX}\right|_{X=x} \mathbb{P}(S = 0|X = x, Z = i). \tag{7}$$

By subtracting (7) from (6) we obtain that

$$\sum_{i=1}^{7} [g_{0i}^{+}(x) - g_{1i}^{-}(x)]\mathbb{P}(S = 0|X = x, Z = i) \leq$$

$$\sum_{i=1}^{7} \mathbb{E}(Y|X = x, Z = i, S = 0)\mathbb{P}(S = 0|X = x, Z = i) \left.\frac{d\mathbb{P}(Z = i|X)}{dX}\right|_{X=x} \leq$$

$$\sum_{i=1}^{7} [g_{1i}^{+}(x) - g_{0i}^{-}(x)]\mathbb{P}(S = 0|X = x, Z = i). \tag{8}$$

Finally, if to the sum of (2), (5), and (8), the identified part of (1) is added, the identification region in Proposition 3.3.1 holds. □

D Proof for the width, $W(x)$, given in 3.7.

Proof. From Equation (3.6), we have that

$$\begin{aligned} W(x) &= \mathbb{E}(D_{1x} - D_{1x}|X=x, S=0)\mathbb{P}(S=0|X=x) \\ &+ \mathbb{E}(f_0^+(X) - f_1^-(X) - (f_1^+(X) - f_0^-(X))|X=x) \\ &+ \sum_{i=1}^{7}(g_{1i}^+(x) - g_{0i}^-(x) - (g_{0i}^+(x) - g_{1i}^-(x)))P(S=0|X=x, Z=i), \end{aligned} \tag{9}$$

where

$$\mathbb{E}(f_0^+(X)-f_1^-(X)-(f_1^+(X)-f_0^-(X))|X=x) = \sum_{i=1}^{7}(f_{0i}^+(x)-f_{1i}^+(x)+f_{0i}^-(x)-f_{1i}^-(x))\mathbb{P}(Z=i|X=x)$$

Note that, on the one hand

$$\begin{aligned} f_{0i}^+(x) - f_{1i}^+(x) &= \sum_{j=1}^{3}(\mathbb{E}(Y|X=x, Z=i, S=j) - y_{0ix}) \left.\frac{d^+\mathbb{P}(S=j|X, Z=i)}{dX}\right|_{X=x} \\ &\quad - (\mathbb{E}(Y|X=x, Z=i, S=j) - y_{1ix}) \left.\frac{d^+\mathbb{P}(S=j|X, Z=i)}{dX}\right|_{X=x} \\ &= \sum_{j=1}^{3}(y_{1ix} - y_{0ix}) \left.\frac{d^+\mathbb{P}(S=j|X, Z=i)}{dX}\right|_{X=x}, \end{aligned}$$

and

$$\begin{aligned} f_{0i}^-(x) - f_{1i}^-(x) &= \sum_{j=1}^{3}(\mathbb{E}(Y|X=x, Z=i, S=j) - y_{0ix}) \left.\frac{d^-\mathbb{P}(S=j|X, Z=i)}{dX}\right|_{X=x} \\ &\quad - (\mathbb{E}(Y|X=x, Z=i, S=j) - y_{1ix}) \left.\frac{d^-\mathbb{P}(S=j|X, Z=i)}{dX}\right|_{X=x} \\ &= \sum_{j=1}^{3}(y_{1ix} - y_{0ix}) \left.\frac{d^-\mathbb{P}(S=j|X, Z=i)}{dX}\right|_{X=x}, \end{aligned}$$

hence

$$\begin{aligned} \mathbb{E}(f_0^+(X) - f_1^-(X) - (f_1^+(X) - f_0^-(X))|X=x) &= \sum_{i=1}^{7}\left(\sum_{j=1}^{3}(y_{1ix} - y_{0ix})\left|\left.\frac{d\mathbb{P}(S=j|X, Z=i)}{dX}\right|_{X=x}\right|\right) \\ &\quad \times\mathbb{P}(Z=i|X=x). \end{aligned} \tag{10}$$

On the other hand,

$$\begin{aligned} g_{1i}^{+}(x) - g_{0i}^{+}(x) &= y_{1ix} \left.\frac{d^{+}}{dX}\mathbb{P}(Z=i|X)\right|_{X=x} - y_{0ix} \left.\frac{d^{+}\mathbb{P}(Z=i|X)}{dX}\right|_{X=x} \\ &= (y_{1ix} - y_{0ix}) \left.\frac{d^{+}\mathbb{P}(Z=i|X)}{dX}\right|_{X=x} \end{aligned}$$

and

$$\begin{aligned} g_{1i}^{-}(x) - g_{0i}^{-}(x) &= y_{1ix} \left.\frac{d^{-}}{dX}\mathbb{P}(Z=i|X)\right|_{X=x} - y_{0ix} \left.\frac{d^{-}}{dX}\mathbb{P}(Z=i|X)\right|_{X=x} \\ &= (y_{1ix} - y_{0ix}) \left.\frac{d^{-}}{dX}\mathbb{P}(Z=i|X)\right|_{X=x}. \end{aligned}$$

Thus,

$$\begin{aligned} \sum_{i=1}^{7}(g_{1i}^{+}(x) - g_{0i}^{-}(x) - (g_{0i}^{+}(x) - g_{1i}^{-}(x)))P(S=0|X=x,Z=i) &= \sum_{i=1}^{7}\left((y_{1ix}-y_{0ix})\left|\frac{d\mathbb{P}(Z=i|X)}{dX}\right|_{X=x}\right) \\ &\quad \times P(S=0|X=x,Z=i). \end{aligned} \tag{11}$$

Applying both (10) and (11) in (9), we have that

$$\begin{aligned} W(x) &= \mathbb{E}(D_{1x} - D_{1x}|X=x,S=0)\mathbb{P}(S=0|X=x) \\ &\quad + \sum_{i=1}^{7}\left(\sum_{j=1}^{3}(y_{1ix}-y_{0ix})\left|\frac{d\mathbb{P}(S=j|X,Z=i)}{dX}\right|_{X=x}\right|\Bigg)\mathbb{P}(Z=i|X=x) \\ &\quad + \sum_{i=1}^{7}\left((y_{1ix}-y_{0ix})\left|\frac{d\mathbb{P}(Z=i|X)}{dX}\right|_{X=x}\right|\Bigg)P(S=0|X=x,Z=i) \\ &= \mathbb{E}(D_{1x} - D_{0x}|X=x,S=0)\mathbb{P}(S=0|X=x) \\ &\quad + \sum_{i=1}^{7}(y_{ix}-y_{0ix})\left[\left|\frac{d\mathbb{P}(Z=i|X)}{dX}\right|_{X=x}\right|\mathbb{P}(S=0|X,Z=i) \\ &\qquad + \sum_{j=1}^{3}\left|\frac{d\mathbb{P}(S=j|X,Z=i)}{dX}\right|_{X=x}\Bigg|\mathbb{P}(Z=i|X=x)\Bigg] \end{aligned}$$

□

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